

# Families of circles on surfaces

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## Abstract

We classify surfaces in 3-space which carry at least 2 families of real circles. Equivalently, we classify surfaces with at least 2 real circles through a generic closed point. We call such surfaces real celestials.

We show that celestials are weak Del Pezzo surfaces. The type of a surface is a tuple  $(d,c)$  defined by the degree of the surface in 3-space and the multiplicity of the absolute conic in the surface. The degree of the Moebius model of a celestial of type  $(d,c)$  is  $2(d-c)$ . The Moebius model of a celestial in the 3-sphere is of degree 2, 4 or 8. We show that the type of a celestial is either  $(1,0)$  (plane),  $(2,1)$  (sphere),  $(2,0)$ ,  $(3,1)$ ,  $(4,2)$  (Darboux cyclides),  $(4,0)$ ,  $(6,2)$ ,  $(7,3)$  or  $(8,4)$ .

Two celestials are Cremona equivalent if their normalizations are diffeomorphic. We classify celestials up to Cremona equivalence. In addition we classify up to diffeomorphism the Moebius model of celestials which are Moebius equivalent to type  $(2,1)$ ,  $(4,2)$  or  $(4,0)$ .

As a result of our classification we obtain an alternative proof for the known fact that a real celestial carries either infinite or at most 6 families of circles.

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# 1 Introduction

## 1.1 Background and motivation

A *celestial* is a surface which is covered by at least two families of circles. This means that through a generic point on the surface there are at least two circles.

In [Schicho \[2001\]](#) surfaces which carry at least two families of conics are classified. In particular it is shown that multiple conical surfaces are algebraic. In this paper we build on that classification.

As a corollary from the classification of celestials in 3-space in this paper we obtain an (alternative) proof for each of the following theorems and conjectures as treated in the literature:

- Blum conjecture: A real celestial has either infinite or at most six families of real circles. This conjecture was posed by [Blum \[1980\]](#) and answered by [Takeuchi \[1987\]](#).
- A surface in 3-space with a line and a circle through each point is either the plane or a quadric surface. This was answered in [Nilov and Skopenkov \[2011\]](#).
- The classification of complex celestials in 3-space. This has been, at least partially, answered by Kummer and Darboux. I was not able to uncover exact references. The families of Villarceau circles were introduced in [Villarceau \[1848\]](#).
- A surface in 3-space which carries three families of circles is a cyclide. This conjecture has been posed in [Pottmann et al. \[2012\]](#).
- The classification of celestials which are cyclides. This has been treated in [Takeuchi \[2000\]](#) from a conformal point of view. We will consider the classification up to diffeomorphism.
- The classification of celestials in 3-space with orthogonal families of circles up to Möbius equivalence. This was answered in [Ivey \[1995\]](#) for smooth surfaces.

There is interest for celestials in architecture and computer aided design: [Pottmann et al. \[2007\]](#) and [Pottmann et al. \[2012\]](#).

## 1.2 Problems

In order to classify celestials, we need to introduce an equivalence relation on surfaces. We start by an informal description of the equivalence relations we consider. Later these definition will be recalled more formally.

The group of Möbius transformations preserve families of Möbius circles. A Möbius circle is either a circle or a line. We can consider the Möbius model of a projective surface, which is contained in a 3-sphere in projective 4-space. In this model the Möbius transformations are automorphisms of the 3-sphere. The *cyclicity* of an embedded surface is defined as the multiplicity of the absolute conic in the surface. The *type* of an embedded surface is defined as a pair of its cyclicity and degree.

- Two surfaces in 3-space are *type equivalent* if and only if they have the same type.

The *Möbius degree* is defined as the degree of the Möbius model of a surface. The Möbius degree for a surface in 3-space is equal to two times its degree minus two times the cyclicity.

A family of circles can uniquely be defined by a divisor class in Picard group of the surface it generates. The number of intersection points of a generic circle in one family with a generic circle in another family is equal to the intersection number of their defining classes. We define the *enhanced Picard group* as the Picard group together with intersection product, a distinguished element (the polarizing divisor class) and the Betti numbers of the divisor classes. In the case of real surfaces we also include a real structure, which is an involution of the enhanced Picard group.

- Two polarized surfaces are *(real) Cremona equivalent* if and only if their (real) enhanced Picard groups are isomorphic.

Anti-canonical models of celestials are Cremona equivalent if and only if they are diffeomorphic.

Now we can finally state the problems we address to in this paper. We will concentrate on real celestials, however our methods can be used to solve the problem for the complex case.

**Problem 1.** Classify real celestials in 3-space up to type equivalence.

The solution to this problem is stated in Theorem 32. We find that celestials in 3-space have Möbius degree either two, four or eight.

**Problem 2.** Classify Möbius models of real celestials in 3-space up to Cremona equivalence.

The solution to this problem is stated in Theorem 36 and Theorem 54. We give an explicit description of the classes in the enhanced Picard group which define the family of circles.

**Problem 3.** Classify Möbius models of real celestials in 3-space of degree less than six up to diffeomorphism.

If the Möbius degree is two then the Möbius model is diffeomorphic to the unit sphere. If the Möbius degree is four then the Möbius model is the anti-canonical model of a weak

degree four Del Pezzo. It follows that Möbius models of degree four are diffeomorphic if and only if they are Cremona equivalent; In this case Theorem 36 presents the solution.

If the Möbius degree is eight and the celestial is Möbius equivalent to a surface of type (4,0) then we give a description of the singular locus up to diffeomorphisms in Theorem 54. The Möbius models are diffeomorphic outside the singular locus, since they are Cremona equivalent.

If the Möbius degree is eight and the real surface is not Möbius equivalent to a surface of type (4,0) then we state properties of the singular locus. However, I did not manage to describe the singular locus up to diffeomorphism (see Remark 60). Such a real surface is of degree six, seven or eight and carries two families of circles with intersection product one. The anti-canonical models (hence unprojections) of the Möbius models of such surfaces are diffeomorphic in  $\mathbf{P}^8$ .

### 1.3 Overview

In section 2 we recall some definitions concerning families and the enhanced Picard group. In section 3 we recall the enhanced Picard group of weak Del Pezzo surfaces and Hirzebruch surfaces. In section 4 we summarize some of the theory concerning divisor classes and real structures of weak Del Pezzo surfaces. In section 5 we recall Möbius geometry. In section 6 we define the equivalence relations as mentioned in the problem section of this introduction more formally. So section 2, 3, 4, 5 and 6 consist of mainly the background which is needed for the following sections. Maybe the reader would prefer to start at section 7 and look back for the definitions and statements as they occur.

In section 7 we start by recalling the classification of multiple conical surfaces from Schicho [2001]. A multiple conical surface is either a weak Del Pezzo surface or a Hirzebruch surface. We will show that a celestial is a weak Del Pezzo surface, and we classify celestials in 3-space up to type equivalence.

In section 8 we shortly discuss the classically known classification of celestials of Möbius degree 2 up to real Cremona equivalence: the plane and the sphere.

In section 9 we start with a summary and some additional observations. Then we recall the classically known classification of celestials of type (2, 0), up to Euclidean equivalence. We classify Möbius models of celestials of Möbius degree 4 up to real Cremona equivalence and diffeomorphism. Finally we state the divisor classes of the families of circles, the singularity configuration and the base points of the families of circles. In particular, we see from this the pairwise intersection numbers of the families.

In section 10 we show examples of celestials of Möbius degree 4.

In section 11 we classify Möbius models of celestials of Möbius degree 8. We start the section by a summary.

In section 12 we show examples of celestials of Möbius degree 8.

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## 2 Families, polarized surfaces and enhanced Picard group

In the following section we recall some notions, which are well known to algebraic geometers, but might be less known to the applied geometry community. It should be noted that some of the definitions below are somewhat non-standard. We want to make explicit what algebraic structure is considered.

**Definition 1. (family of curves)** Let  $Z$  be a complex projective surface. Let  $I$  be a non-singular curve. A *family* of  $Z$  indexed by  $I$  is defined as

$$F \subset I \times Z$$

where

- $F$  is an irreducible algebraic subset of codimension one, and
- the second projection  $F \xrightarrow{\pi_Z} Z$  is dominant.

The *family members* of  $F$  are defined as

$$F_i = (\pi_Z \circ \pi_I^{-1})(i)$$

for all  $i \in I$  where

- $F \xrightarrow{\pi_Z} Z$  and  $F \xrightarrow{\pi_I} I$  are the projection maps.

**Definition 2. (Picard group)** Let  $X$  be a non-singular complex projective surface. The *group of divisors* of  $X$  is the additive group generated by the irreducible curves on  $X$ . Divisors  $D$  and  $D'$  are *linear equivalent* if and only if there exists a divisor  $B$  such that  $D + B$  and  $D' + B$  belong to a family of  $X$  indexed by  $\mathbf{P}^1$ . The *Picard group*  $\text{Pic}X$  is the group of divisors modulo linear equivalence.

**Definition 3. (polarized surface)** A *polarized surface* is defined as a pair  $(X, D)$  where

- $X$  is a non-singular projective surface, and
- $D$  in  $\text{Pic}(X)$ .

A *projected polarized surface* is defined as a triple  $(X, D, V^n)$  such that  $(X, D)$  is a polarized surfaces and  $V^n \subset \mathbf{P}(H^0(X, D))$  is a projective vector subspace of dimension  $n > 0$ . The *projected model* of  $(X, D, V^n)$  is defined as  $Z := \varphi_{V^n}(X) \subset \mathbf{P}^n$ . Note that a projected polarized surface induces the following diagram:



$$\begin{array}{ccc}
X & \xrightarrow{\varphi_D} & \varphi_D(X) \subset \mathbf{P}^{h^0(D)-1} \\
\varphi_V \downarrow & \swarrow \pi & \\
Z \subset \mathbf{P}^n & & 
\end{array}$$

where  $h^0(D)$  is the dimension of  $H^0(X, D)$  and  $\pi$  is a linear projection with center outside the surface such that the diagram commutes.

**Definition 4. (enhanced Picard group)** Let  $Y = (X, D, V^n)$  be a projected polarized surface. The *enhanced Picard group*  $A(Y)$  is defined as

$$(\text{Pic}(X), D, \cdot, h)$$

where

- $\text{Pic}(X)$  is the Picard group,
- $\text{Pic}(X) \times \text{Pic}(X) \rightarrow \mathbf{Z}$  is the intersection product on divisor classes, and
- $\mathbf{Z} \times \text{Pic}(X) \xrightarrow{h} \mathbf{Z}_{\geq 0}$  assigns the  $i$ -th Betti number to a divisor class for  $i \in \mathbf{Z}$ .

For  $h(i, D)$  we use the notation  $h^i(D)$ .

**Definition 5. (unprojected class, strict unprojected class)** Let  $Y = (X, D, V^n)$  be a projected polarized surface. Let  $Z \subset \mathbf{P}^n$  be the projected model of  $Y$ . Let  $A(Y) = (\text{Pic}(X), D, \cdot, h)$  be the enhanced Picard group. Let  $X \xrightarrow{\varphi} Z$  be the map associated to  $V^n$  (which is defined everywhere). Let  $C \subset Z$  be a subvariety of  $Z$ . Let  $C' := \varphi^{-1}(C)$  be the preimage of  $C$ . The *unprojected class* of  $C$  is defined as

- the divisor class of  $C'$  in  $A(Y)$  if  $C'$  is a curve, and
- $0 \in A(Y)$  otherwise.

Let  $F'$  be the union of the components of  $C'$  such that  $\varphi(F')$  is a point. Let  $B'$  be  $C' - F'$ . The *strict unprojected class* of  $C$  is defined as a pair

$$(B, F)$$

where  $B$  is the divisor class of  $B'$  and  $F$  is the divisor class of  $F'$ .

**Definition 6. (real projected polarized surface)** Let  $(X, D, V^n)$  be a projected polarized surface. Let  $X \xrightarrow{\sigma} X$  be the complex conjugation (thus  $\text{Gal}(\mathbf{C}|\mathbf{R}) = \langle \sigma \rangle$ ). We call  $Y = (X, D, V^n, \sigma)$  a *real projected polarized surface*.

**Proposition 7. (enhanced Picard group of a real surface)**

Let  $(X, D, V^n, \sigma)$  be a real projected polarized surface. Let  $A(Y) = (\text{Pic}(X), D, \cdot, h)$  be the enhanced Picard group.

We have that  $\sigma$  induces an automorphism on  $A(Y)$ .

**Proof:** See Silhol [1989]. 

**Definition 8. (real structure and real enhanced Picard group)** Let  $Y = (X, D, V^n, \sigma)$  be a real projected polarized surface. Let  $A(Y) = (\text{Pic}(X), D, \cdot, h)$  be the enhanced Picard group. A *real structure* on  $Y$  is defined as the automorphism  $A(Y) \xrightarrow{\sigma'} A(Y)$  induced by  $\sigma$ . Let  $A(Y) = (\text{Pic}(X), D, \cdot, h)$  be the enhanced Picard group. The *real enhanced Picard group* is defined as  $(\text{Pic}(X), D, \cdot, h, \sigma')$ . By abuse of notation we denote  $\sigma'$  also by  $\sigma$ .

**Example 9. (polarizations of the projective plane)**

Curves of the same degree in  $\mathbf{P}^2$  are contained in a family over  $\mathbf{P}^1$ .

It follows that the Picard group is

$$\text{Pic}(\mathbf{P}^2) = \mathbf{Z}\langle L \rangle$$

where  $dL$  is the linear equivalence class of a degree  $d$  curve in the projective plane.

We have that  $H^0(\mathbf{P}^2, 2L) = \mathbf{C}\langle x^2, y^2, z^2, xy, xz, yz \rangle$ .

The polarized surface  $Y := (\mathbf{P}^2, 2L)$  is called the Veronese surface.

The intersection product on  $\text{Pic}(\mathbf{P}^2)$ :  $L \cdot L = 1$ . Using Riemann Roch theorem we find that  $h^0(dL) = \frac{1}{2}d(d+3) + 1$  and  $h^i(dL) = 0$  for  $i > 0$  for  $d \geq 0$  and  $h^i(dL) = 0$  for all  $i$  if  $d < 0$ . Note that we have a complete description of the enhanced Picard group  $A(Y) := (\mathbf{Z}\langle L \rangle, 2L, \cdot, h)$ .

Since the Picard group is generated by one class, there is only one possible real structure:  $\sigma$  equals the identity. Let  $V^3 = \mathbf{P}(\langle xy, xz, yz, z^2 \rangle) \subset \mathbf{P}(H^0(\mathbf{P}^2, 2L))$

The real projected polarized surface  $(\mathbf{P}^2, 2L, V^3, \sigma)$  is called the real Roman surface.

The polarizing divisor is  $2L$ . Lines in  $\mathbf{P}^2$  are embedded as conics in  $\mathbf{P}^5 = \mathbf{P}(V^5)$  since  $2L \cdot L = 2$ . The degree of its projected model equals  $(2L)^2 = 4$ .

### 3 Enhanced Picard group of weak Del Pezzo and Hirzebruch surfaces

In this section we recall the enhanced Picard groups of Del Pezzo surfaces and Hirzebruch surfaces in terms of unimodular lattices and the anti-canonical divisor class. References are Dolgachev [2012] and Beauville [1983].

**Definition 10. (weak Del Pezzo and Hirzebruch surfaces)** Let  $Y = (X, D, V^n)$  be a projected polarized surface. We call  $Y$  a *Hirzebruch surface* if and only if  $X \xrightarrow{\varphi_K} \mathbf{P}^1$  is a geometrically ruled surface over  $\mathbf{P}^1$  such that either  $F = aD$ , or  $F = a(2D + K)$  for some  $a \in \mathbf{Z}_{>0}$ . Note that  $X$  is rational in this case. We call  $Y$  a *weak Del Pezzo surface* if and only if  $-K$  is nef and big, and either  $D = -K$ ,  $D = -\frac{1}{2}K$ ,  $D = -\frac{1}{3}K$ , or  $D = -\frac{2}{3}K$ . Let  $Y' = (X', D', V'^n)$  be a weak Del Pezzo surface. We call  $\varphi_{D'}(X')$  the *anti-canonical model* of  $Y'$ .

**Definition 11. (standard bases)** The *standard Del Pezzo basis*  $B(r)$  is defined as a unimodular lattice

$$B(r) = \mathbf{Z}\langle H, Q_1, \dots, Q_r \rangle,$$

with inner product  $H^2 = 1$ ,  $Q_i Q_j = -\delta_{ij}$  and  $HQ_i = 0$ . The *standard geometrically ruled basis*  $P(r)$  is defined as a unimodular lattice

$$P(r) = \mathbf{Z}\langle H, F \rangle,$$

with inner product  $F^2 = 0$ ,  $FH = 1$  and  $H^2 = r$ .


**Proposition 12. (enhanced Picard group of Del Pezzo surfaces)**

Let  $(X, D, V^n)$  be a weak Del Pezzo surface. Let  $A(Y) = (\text{Pic}(X), D, \cdot, h)$  be the enhanced Picard group of  $X$ . Let  $K$  be the canonical class of  $X$ . Let  $B(r) = \mathbf{Z}\langle H, Q_1, \dots, Q_r \rangle$  be the standard Del Pezzo basis.

a) If  $K^2 \neq 8$  then  $A(Y) \cong B(9 - K^2)$  with  $-K = 3H - Q_1 - \dots - Q_r$ .

Let  $P(r) = \mathbf{Z}\langle H, F \rangle$  be the standard geometrically ruled basis.


b) If  $K^2 = 8$  then  $A(Y) \cong P(r)$  with  $-K = 2H - (r - 2)F$  for  $r = 0, 1$  or  $2$ .

**Proof:** See appendix E, section 2 of [Lubbes \[2011\]](#) or [Dolgachev \[2012\]](#). 

**Proposition 13. (enhanced Picard group of Hirzebruch surfaces)**

Let  $(X, D, V^n)$  be a Hirzebruch surface. Let  $A(Y) = (\text{Pic}(X), D, \cdot, h)$  be the enhanced Picard group of  $X$ . Let  $K$  be the canonical class of  $X$ . Let  $P(r) = \mathbf{Z}\langle H, F \rangle$  be the standard geometrically ruled basis.

a) We have that  $A(Y) \cong P(r)$  and  $-K = 2H - (r - 2)F$  for some  $r \geq 0$ .

**Proof:** See [Beauville \[1983\]](#), chapter 3, proposition 18, page 34, and chapter 4, proposition 1, page 40. 

## 4 Weak Del Pezzo surfaces

**Definition 14. (effective Del Pezzo zero-set and Dynkin type)** Let  $Y = (X, D, V^n)$  be a weak Del Pezzo surface. Let  $A(Y) = (\text{Pic}(X), D, \cdot, h)$  be the enhanced Picard group. The *effective Del Pezzo zero-set*  $F(Y)$  is defined as

$$\{ C \in A(Y) \mid C^2 = -2, DC = 0 \text{ and } h^0(C) > 0 \}$$

Let  $F(Y)$  be the effective Del Pezzo zero-set. Let  $B(r) = \mathbf{Z}\langle H, Q_1, \dots, Q_r \rangle$  be the standard Del Pezzo basis. We use the following notation for classes in  $F(Y)$ :

- $Q_1 - Q_2$  is 12,
- $H - Q_1 - Q_2 - Q_3$  is 1123,
- $2H - Q_1 - Q_2 - Q_3 - Q_4 - Q_5 - Q_6$  is 278, where 7 are 8 the indices of the omitted  $Q_i$ , and
- $3H - 2Q_1 - Q_2 - Q_3 - Q_4 - Q_5 - Q_6 - Q_7 - Q_8$  is 301 where 1 is the index of the  $Q_i$  which has coefficient two.

Up to permutation of the  $Q_i$  in the standard Del Pezzo basis we represented all possible elements in  $F(Y)$  (see [Lubbes \[2012a\]](#) where these are called C1 label elements). The *intersection diagram* of  $F(Y)$  is defined as the graph obtained by connecting two elements in  $F(Y)$  if and only if their intersection product is non-zero. The *Dynkin type* of  $F(Y)$  is the Dynkin type of its intersection diagram (if it is a Dynkin diagram).

**Definition 15. (irreducible Del Pezzo one-set)** Let  $Y = (X, D, V^n, \sigma)$  be a real weak Del Pezzo surface. Let  $A(Y) = (\text{Pic}(X), D, \cdot, h)$  be the enhanced Picard group. Let  $F(Y)$  be the effective Del Pezzo zero-set. The *irreducible Del Pezzo one-set* is defined as

$$E(Y) = \{ E \in A(Y) \mid E^2 = -1, DE = 1 \text{ and } EF > 0 \text{ for all } F \in F(Y) \}$$

The *real irreducible Del Pezzo one-set* is defined as

$$E_{\mathbf{R}}(Y) = \{ E \in E(Y) \mid \sigma(E) = E \}.$$

**Definition 16. (irreducible Del Pezzo two-set)** Let  $Y = (X, D, V^n, \sigma)$  be a real weak Del Pezzo surface. Let  $A(Y) = (\text{Pic}(X), D, \cdot, h)$  be the enhanced Picard group. Let  $F(Y)$  be the effective Del Pezzo zero-set. The *irreducible Del Pezzo two-set* is defined as

$$G(Y) = \{ G \in A(Y) \mid G^2 = 0, DG = 2 \text{ and } GF > 0 \text{ for all } F \in F(Y) \}.$$

The *real irreducible Del Pezzo two-set* is defined as

$$G_{\mathbf{R}}(Y) = \{ G \in G(Y) \mid \sigma(G) = G \}.$$

Let  $B(5) = \mathbf{Z}\langle H, Q_1, \dots, Q_5 \rangle$  be the standard Del Pezzo basis. If  $Y$  is a weak Del Pezzo surface of degree 4 then we use the following notation for classes in  $G(Y)$  and  $G_{\mathbf{R}}(Y)$ :

- $H - Q_1$  is  $a1$ , and
- $H - Q_2 - Q_3 - Q_4 - Q_5$  is  $b1$ , where 1 is the index of the omitted  $Q_i$ .

Up to permutation of the  $Q_i$  in the standard Del Pezzo basis we represented all possible elements in  $G(Y)$  and  $G_{\mathbf{R}}(Y)$ . Note that  $ai \cdot ai = 0$ ,  $ai \cdot bi = 2$  and  $ai \cdot bj = ai \cdot aj = 1$  for all  $i \neq j$  in  $[1, 5]$ .

**Proposition 17. (classes of weak Del Pezzo surfaces)**

Let  $Y = (X, D, V^n, \sigma)$  be a real weak Del Pezzo surface. Let  $F(Y)$  be the effective Del Pezzo zero-set.

**a)** The singular points on  $\varphi_D(X)$  are double points. The unprojected class of a double point is a connected component of the intersection diagram of  $F(Y)$ . The Dynkin type of this connected component defines the Dynkin type of the corresponding singularity. Conversely, each class in  $F(Y)$  corresponds to a unique curve on  $X$ , which is contracted to a singular double point. The singular point is real if and only if the corresponding connected component is sent to itself by the real structure  $A(Y) \xrightarrow{\sigma} A(Y)$ .

Let  $E(Y)$  be the irreducible Del Pezzo one-set.


**b)** Each class in  $E(Y)$  is the class of a line on  $Y$ .

Let  $G(Y)$  be the (real) irreducible Del Pezzo two-set.

**c)** Each class in  $G(Y)$  uniquely defines a one dimensional family of (real) conics of  $Y$ . Such a family can be defined by the fibers of the morphism associated to the class. The class of a family is equal to the unprojected class of a generic circle in the family.

Let  $Z$  be the projected model of  $Y$ .

**d)** A family of conics defined by  $G \in \mathbf{R}(Y)$  has a base point on  $Z$  if and only if it has nonzero intersection with a class in a connected component of the intersection diagram of  $F(Y)$ . The base point is the singular double point corresponding to the connected component.

**Proof:** See [Lubbes \[2012a\]](#) or [Dolgachev \[2012\]](#). For **a)** note that by definition  $D = -K$  is the anti-canonical divisor class. The curves in  $F(Y)$  are zero against  $-K$  and thus are contracted to singular points by  $\varphi_D$ . For **c)** and **d)** see also [Schicho \[2001\]](#) and [Lubbes \[2012b\]](#). 

**Proposition 18. (effective Del Pezzo zero-set of weak Del Pezzo surfaces of degree four)**

Let  $Y = (X, D, V^n, \sigma)$  be a real weak Del Pezzo surface of degree 4. Let  $F(Y)$  be the effective Del Pezzo zero-set.

**a)** Up to Cremona isomorphism  $F(Y)$  is in Table [19](#).

Let  $G_{\mathbf{R}}(Y)$  be the real irreducible Del Pezzo two-set.


**b)** Up to Cremona isomorphism the cardinality of  $G_{\mathbf{R}}(Y)$  is in Table 21.

Let  $A(Y) = (\text{Pic}(X), D, \cdot, h, \sigma)$  be the real enhanced Picard group.

**c)** Up to Cremona equivalence  $A(Y) \xrightarrow{\sigma} A(Y)$  is in Table 20.

**d)** We have that  $\varphi_D(X)$  has no real lines if and only if RI equals 13, 14 or 15 in Table 20.

**e)** If RI equals 15 then  $\varphi_D(X)$  consist of two real components, and one real component otherwise.

**Proof:** See Lubbes [2012a]. For **d)** we recall that minus one classes are defined as classes with self intersection minus one, and intersection one with the anti-canonical divisor class. The real minus one classes are fixed by the real structure. The real minus one classes which are positive against  $F(Y)$  correspond to lines on the projected model. We have that  $F(Y)$  is closed under a real structure. For **e)** we note that the real structures  $RI = 10, \dots, 15$  have respectively Dynkin types  $A_0, A_1, 2A_1, 2A_1, 3A_1$  and  $D_4$ . See Wall [1987], section 3, corollary 2, page 57 and lemma 5, page 59. 

**Table 19. (effective Del Pezzo zero-set of weak Del Pezzo surfaces of degree four)**

- The CI column assigns an index to each row for future reference.
- The  $F(Y)$  column denotes the effective Del Pezzo zero-set.
- The Dynkin column denotes the Dynkin type.

CI	$F(Y)$	Dynkin
16		$A_0$
17	45	$A_1$
18	23, 45	$2A_1$
19	1123, 45	$2A_1$
20	34, 45	$A_2$
21	1123, 23, 45	$3A_1$
22	12, 34, 45	$A_2 + A_1$
23	23, 34, 45	$A_3$
24	1123, 34, 45	$A_3$
25	1145, 1123, 23, 45	$4A_1$
26	1123, 12, 23, 45	$A_2 + 2A_1$
27	1123, 12, 34, 45	$A_3 + A_1$
28	12, 23, 34, 45	$A_4$
29	1123, 23, 34, 45	$D_4$
30	1145, 1123, 12, 23, 45	$A_3 + 2A_1$
31	1123, 12, 23, 34, 45	$D_5$

**Table 20. (real structures of weak Del Pezzo surfaces of degree four)**

- Let  $Y = (X, D, V^n, \sigma)$  be a real weak Del Pezzo surface of degree 4.
- Let  $A(Y) = (\text{Pic}(X), D, \cdot, h, \sigma)$  be the real enhanced Picard group.
- We give explicit coordinate descriptions of real structures on weak Del Pezzo surfaces of degree 4:

$$\sigma : A(Y) \rightarrow A(Y), \quad (H, Q_1, \dots, Q_5) \mapsto (D_0, \dots, D_5)$$

The coordinates are with respect to the standard Del Pezzo basis  $B(5)$ .

- We assign an index to each real structure, which we will refer to as  $RI$ . The indices are in accordance with the indices in [Lubbes \[2012a\]](#).

Below we denote  $(D_0, \dots, D_5)$ :

- $RI = 10$ :  $(H, Q_1, Q_2, Q_3, Q_4, Q_5)$ ;
- $RI = 11$ :  $(H, Q_1, Q_2, Q_3, Q_5, Q_4)$ ;
- $RI = 12$ :  $(H, Q_1, Q_3, Q_2, Q_5, Q_4)$ ;
- $RI = 13$ :  $(2H - Q_1 - Q_2 - Q_3, H - Q_2 - Q_3, H - Q_1 - Q_3, H - Q_1 - Q_2, Q_5, Q_4)$ ;
- $RI = 14$ :  $(2H - Q_1 - Q_2 - Q_3, H - Q_2 - Q_3, H - Q_1 - Q_2, H - Q_1 - Q_3, Q_5, Q_4)$ ;
- $RI = 15$ :  $(3H - 2Q_1 - Q_2 - Q_3 - Q_4 - Q_5, 2H - Q_1 - Q_2 - Q_3 - Q_4 - Q_5, H - Q_1 - Q_2, H - Q_1 - Q_3, H - Q_1 - Q_4, H - Q_1 - Q_5)$ ;

**Table 21. (real irreducible Del Pezzo two-set of weak real Del Pezzo surfaces of degree four)**

- Let  $Y = (X, D, V^n, \sigma)$  be a real weak Del Pezzo surface of degree 4.
- Let  $F(Y)$  be the effective Del Pezzo zero-set.
- The Dynkin column denotes the Dynkin type of  $F(Y)$ .
- The CI column denotes the index for  $F(Y)$  in [Table 19](#).
- The RI row denote the index for  $\sigma$  in [Table 20](#).
- The indices are in accordance with [Lubbes \[2012a\]](#).
- Let  $G_{\mathbf{R}}(Y)$  be the real irreducible Del Pezzo two-set.
- Each entry in the table denotes the possible cardinalities of  $G_{\mathbf{R}}(Y)$ .

Dynkin	$CI \downarrow   RI \rightarrow$	10	11	12	13	14	15
A0	16	10	6	2	6	2	2
A1	17	8	4, 6	2	4	2	
2A1	18	6	4	2			
2A1	19	7	3		3, 5	1	1
A2	20	6	2		2	2	
3A1	21	5	3		3	1	
A2 + A1	22	4	2				
A3	23	4		2			
A3	24	5	1		1, 3		
4A1	25	4		2	2, 4		
A2 + 2A1	26	3				1	
A3 + A1	27	3	1				
A4	28	2					
D4	29	3			1		
A3 + 2A1	30	2			2		
D5	31	1					

## 5 Möbius geometry

We recall 3 dimensional projective model for Möbius geometry and include some proofs for the convenience of the reader. This section is well known to the applied geometers, but might be less known to the algebraic geometry community. Generalization of the propositions and proofs in this section to higher dimensional Möbius geometry is straightforward.

**Definition 22. (absolute variety, Möbius circle)** The *absolute variety* is defined as

$$A : x_1^2 + \dots + x_n^2 = x_0 = 0 \subset \mathbf{P}^n.$$

If  $n = 3$  then we call  $A$  the *absolute conic*. A *circle* in projective space, is a conic which intersect the absolute variety in two distinct points. A *Möbius circle* is either a circle or a line.

**Definition 23. (group of Möbius transformations of projective space)** The *group of Euclidean transformations with dilations* of  $\mathbf{P}^n$  are the projective transformations, which preserve the absolute conic. The *inversion transformation* in  $\mathbf{P}^n$  is defined as follows. First we consider an  $n$ -sphere  $S^n$  of radius  $r$  with center  $O$ . The inversion transformation of a point  $P$  with respect to  $S^n$  is a point  $Q$  on the line through  $O$  and  $P$  such that  $P$  lies in the middle. Furthermore we have that  $OP \times OQ = r^2$ . The *group of Möbius transformations* of  $\mathbf{P}^n$  is generated by the Euclidean transformations with dilations and inversions.



**Example 24. (circle and inversions)**

The homogeneous equation of a circle with center at  $(1 : a : b)$  and radius  $r$  is

$$(x_1 - ax_0)^2 + (x_2 - bx_0)^2 - (rx_0)^2 = 0$$

and intersects the absolute variety  $A : x_1^2 + x_2^2 = x_0 = 0$  in two distinct points  $(0 : 1 : \pm i)$ .

In the Euclidean plane, inversion with respect to the unit circle  $C$  with center  $O$  sends a point  $P$  to a point  $Q$  such that  $OP \times OQ = 1$  and  $OPQ$  lie on a line. The inversion with respect to  $C$  of a circle with radius two and center on  $C$  is a line tangent to  $C$ . Stereographic projections are special cases of inversions.

The inversion transformation with respect to the unit  $n$ -sphere centered at  $(0 : \dots : 0 : 1)$  is:

$$x \mapsto (x_1x_0 : \dots : x_nx_0 : x_1^2 + \dots + x_n^2).$$

**Definition 25. (Möbius transformation diagram)** The *Möbius transformation diagram* is defined as:

$$\text{MTD}(\mu, f, \beta) := \begin{array}{ccccc} S^3 & & \xrightarrow{\beta} & & S^3 \\ f^{-1} \uparrow & & \circlearrowleft & & \downarrow f \\ \mathbf{P}^3 & & \xrightarrow{\mu} & & \mathbf{P}^3 \end{array}$$

where

- $f$  is the stereographic projection from the point  $\infty$  and
- $\beta$  is a projective isomorphism which preserves the *Möbius 3-sphere*  $S^3 \subset \mathbf{P}^4$ .

In this paper we make the following choice of coordinates:

- $S^3 : a^2 + b^2 + c^2 + d^2 - e^2 = 0$ , and
- $f : (a : b : c : d : e) \mapsto (a : b : c : e - d)$ .

**Definition 26. (multiple conical surfaces, celestials, type, Möbius model and Möbius degree)** Let  $Y = (X, D, V^n)$  be a projected polarized surface. Let  $Z \subset \mathbf{P}^n$  be the projected model of  $Y$ . We call  $Y$  a *multiple conical surface* if and only if  $Z$  has at least 2 families of conics. We call  $Y$  a *celestial* if and only if  $Z$  has at least 2 families of Möbius circles. We call  $Y$  of *type*  $(d, c)$  if and only if  $Z$  is of degree  $d$  and  $Z$  contains the absolute conic with multiplicity  $c$ . Let  $\text{MTD}(\mu, f, \beta)$  be the Möbius transformation diagram. The *Möbius model*  $M$  of  $Y$  is defined as  $\overline{f^{-1}(Z)}$ . The *Möbius degree* of  $Y$  is defined as the degree of its Möbius model.

We recall the following well know proposition due to Möbius. We include a proof for the three dimensional case for the convenience of the reader. The generalization to any dimension is straightforward. The two dimensional analogue of the following proposition is illustrated very nicely in [Arnold and Rogness \[2008\]](#).

**Proposition 27. (factorization of Möbius transformations)**

Let  $\text{MTD}(\mu, f, \beta)$  be the Möbius transformation diagram.

**a)** We have that  $f^{-1} : \mathbf{P}^3 \rightarrow S^3$ ,  $(x : y : z : w) \mapsto (2wx : 2wy : 2wz : x^2 + y^2 + z^2 - w^2 : x^2 + y^2 + z^2 + w^2)$ .

**b)** For any Möbius transformation  $\mu$  there exists a linear isomorphism  $S^3 \xrightarrow{\beta} S^3$  such that the Möbius transformation diagram commutes.

**Proof:**

*Claim 1:* We have that **a**).

We have that  $f : (a : b : c : d : e) \mapsto (a : b : c : e - d) = (x : y : z : w)$ . From  $w = e - d$  it follows that  $w(e + d) = e^2 - d^2$ . We have that  $2d = e + d - w$  and  $2e = e + d + w$ . From  $a^2 + b^2 + c^2 + d^2 - e^2 = 0$  it follows that  $e + d = \frac{x^2 + y^2 + z^2}{w}$  and that this claim holds..

*Claim 2:* If  $\mu$  is a translation with dilation then **b**) holds.

We have that  $\mu : (x : y : z : w) \mapsto (x + x_t w : y + y_t w : z + z_t w : s w)$  where  $s$  is the dilation factor and  $(x_t : y_t : z_t)$  the translation vector. It follows that  $\beta := f^{-1} \circ \mu \circ f : (a : b : c : d : e) \mapsto (2s(a - dx_t + ex_t) : 2s(b - dy_t + ey_t) : 2s(c - dz_t + ez_t) : s^2((e + d) + (e - d)(x_t^2 + y_t^2 + z_t^2) + 2(ax_t + by_t + cz_t)) - (e - d) : s^2((e + d) + (e - d)(x_t^2 + y_t^2 + z_t^2) + 2(ax_t + by_t + cz_t)) + (e - d))$  is linear using the relation  $a^2 + b^2 + c^2 + d^2 - e^2 = 0$ .

*Claim 3:* If  $\mu$  is a rotation along the origin then **b**).

We have that  $\beta$  consists of rotations of  $S^3$  along the axes  $a = d = 0$ ,  $b = d = 0$  and  $c = d = 0$  such that the plane at infinity  $e = 0$  is pointwise fixed.

*Claim 4:* If  $\mu$  is an inversion transformation then **b**).

We have that  $\mu : x \mapsto (xw : yw : zw : x^2 + y^2 + z^2)$ . We choose  $\beta : (a : b : c : d : e) \mapsto (a : b : c : -d : e)$ . This claim follows from claim 1).

*Claim 5:* We have that **b**).

The group of Möbius transformations is generated by the Euclidean transformations with dilations and the inversion with respect to unit sphere. This claim follows from claim 2), claim 3) and claim 4).

We remark that the embedding of a sphere is associated to a unique divisor class. From this it follows that the polynomial isomorphisms of spheres are linear. The Euclidean transformations with dilations correspond to isomorphisms of  $S^3$  fixing the center of projection.



**Proposition 28. (Möbius degree)**

Let  $W \subset \mathbf{P}^3$  be a variety.

- a) The Möbius degree of  $W$  is equal to the number of intersection points with  $\dim(W)$  generic spheres, outside the absolute conic.
- b) If  $W$  is a surface of type  $(d, c)$  then the Möbius degree of  $W$  equals  $2(d - c)$ .

**Proof:** Let  $\text{MTD}(\mu, f, \beta)$  be the Möbius transformation diagram. Let  $M := f^{-1}(W)$  be the Möbius model of  $W$ . Let  $\Gamma$  be the linear series associated to  $\mathbf{P}^3 \xrightarrow{f^{-1}} S^3$ .

*Claim 1:* We have that **a)**.

We have that  $\Gamma$  is a four dimensional linear series of spheres in  $\mathbf{P}^3$  with the absolute conic as base locus. We recall from [Hartshorne \[1977\]](#), chapter 1, section 7, page 48, that the degree of  $f^{-1}(W)$  is defined as the cardinality of  $f^{-1}(Z)$  intersected with  $\dim W$  generic hyperplane sections. A hyperplane section pulls back along  $f^{-1}$  to a sphere in  $\Gamma$ . We have that  $f^{-1}$  is not defined at the absolute conic. It follows that this claim holds.

*Claim 2:* We have that **b)**.

Two generic spheres in  $\Gamma$  intersect in a quartic curve, consisting of a circle  $C$  and the absolute conic  $A$ . From claim 1) it follows that  $\deg f^{-1}(Z)$  is equal to  $\#(C \cap Z - A)$ . A circle intersects the absolute conic  $A$  in 2 points. It follows that  $C$  intersects  $Z \cdot A$  with multiplicity  $2c$ . We have that  $C$  intersects  $Z$  in  $2d$  points. It follows that the number of intersections outside the absolute conic is equal to  $2d - 2c$ . ☕

**Proposition 29. (type)** Let  $Z \subset \mathbf{P}^3$  be a surface of type  $(d, c)$ .

a) We have that  $2(d - c)$  is invariant under Möbius transformations.

b) There exists a Möbius transformation  $\mathbf{P}^3 \xrightarrow{\mu} \mathbf{P}^3$  such that  $\mu(Z)$  is of any type

$$(2(d - c) - m, (d - c) - m)$$

where  $m$  is the multiplicity of a finite point outside the absolute conic.

c) The number of families of Möbius circles on  $Z$  is invariant under Möbius transformations.

**Proof:**

*Claim 1:* We have that a).

Half the Möbius degree is invariant under applying  $\beta$ . This claim follows from Proposition 28.

*Claim 2:* We have that b).

If  $\text{mult}_\infty(\beta \circ f^{-1})(Z) = m$  at the center of projection, then  $f$  reduces the degree by  $m$ . This claim follows from claim 2).

Let  $\text{MTD}(\mu, f, \beta)$  be the Möbius transformation diagram.

*Claim 3:* We have that c).

From Proposition 27 it follows that  $\mu = f \circ \beta \circ f^{-1}$ . From Proposition 28 it follows that  $f^{-1}$  maps circles to conics in  $S^3$ , which are circles. We have that  $\beta$  rotates  $S^3$  and thus preserves circles. The preimage of the absolute conic under  $f$  is the surface  $B : a^2 + b^2 + c^2 = e - d = 0 \subset \mathbf{P}^4$ . Thus  $B$  is the section of  $S^3$  with the hyperplane  $e - d = 0$ . By definition the degree of a curve in  $\mathbf{P}^n$  is defined by the number of intersections with a generic hyperplane. It follows that the conic  $C$  in  $S^3 \subset \mathbf{P}^4$  intersects a hyperplane  $e - d = 0$  in 2 points. It follows that  $f(C)$  intersects  $A = f(B)$  in two points, and thus is a circle. If a circle goes through the center of projection  $(0 : 0 : 0 : 1 : 1)$  then it is projected to a line. From  $f^{-1}, \beta$  and  $f$  preserving Möbius circles it follows that this claim holds. ☕

## 6 Equivalence relations on polarized surfaces

**Definition 30. (equivalence relations of polarized surfaces)** We define projected polarized surfaces to be *type equivalent* if and only if their projected models are of the same type. We define projected polarized surfaces to be *Cremona equivalent* if and only if their enhanced Picard groups are isomorphic. We define projected polarized surfaces to be *real Cremona equivalent* if and only if their real enhanced Picard groups are isomorphic. We define projected polarized surfaces to be *Euclidean equivalent* if and only if their projected models are sent to each other by an element from the group of Euclidean transformations with dilations. We define projected polarized surfaces to be *Möbius equivalent* if and only if their projected models are sent to each other by an element from the group of Möbius transformations.

## 7 Classification of celestials up to type equivalence

### Theorem 31. ((Schicho 2001) multiple conical surfaces)

Let  $Y = (X, D, V^n)$  be a multiple conical surface. Let  $K$  be the canonical divisor class of  $Y$ . Let  $B(r) = \mathbf{Z}\langle H, Q_1, \dots, Q_r \rangle$  with  $r = 9 - K^2$  be the standard Del Pezzo basis. Let  $P(s) = \mathbf{Z}\langle H, F \rangle$  be the standard geometrically ruled basis.

a) We have one of the following cases:

	$K^2$	name	Pic	$D$	$D^2$	$n$	classes	dim	description
1	9	DP	$B(0)$	$H$	1	2	$2H$	5	plane
2	9	DP	$B(0)$	$2H$	4	5	$H$	2	Veronese surface
3	8	DP	$P(0)$	$H + F$	2	3	$H + F$	3	saddle
4	8	DP	$P(0)$	$2H + 2F$	8	8	$H, F$	1	smooth
5	8	DP	$P(2)$	$H$	2	3	$H$	3	singular cone
6	$3, \dots, 7$	DP	$B(r)$	$3H - Q_1 - \dots - Q_r$	$9 - r$	$9 - r$	$\dagger$	1	normal
7	8	HZ	$P(0)$	$H + 2F$	4	5	$H$	1	ruled by lines
8	0	HZ	$P(1)$	$H + F$	3	4	$H$	2	ruled by lines

where

- the ‘name’ column denotes weak Del Pezzo surface (DP) or Hirzebruch surface (HZ),
- the ‘Pic’ column denotes a basis for the enhanced Picard group of the surface,
- the ‘ $D$ ’ column defines  $D$  in terms of the given basis,
- the ‘ $n$ ’ column denotes the embedding dimension of  $\varphi_D(X) \subset \mathbf{P}^n$ ,
- the ‘classes’ column denotes the divisor classes of the conical families in the enhanced Picard group,
- the ‘dim’ column gives the projective dimension of the conical family classes,
- the ‘description’ column describes the singularities of  $\varphi_D(X)$  (thus not of the projected model of  $Y$ ),

and  $\dagger$  stands for the set of classes which up to permutation of the  $Q_i$  in the standard Del Pezzo basis are of the form  $H - Q_1, 2H - Q_1 - Q_2 - Q_3 - Q_4$  or  $3H - 2Q_1 - Q_2 - \dots - Q_6$ , and which can not be written as the sum of two effective classes (this is called the irreducible Del Pezzo two set in [Lubbes \[2012a\]](#)).

**Proof:** See [Schicho \[2001\]](#). In [Lubbes \[2012b\]](#) there is an alternative proof and a classification up to Cremona equivalence. The conical families on weak Del Pezzo surfaces of degree  $3, \dots, 7$  are of minimal degree. ☕

**Theorem 32. (classification of celestials in 3-space up to type equivalence)**

Let  $Y = (X, D, V^3)$  be a celestial.

a) We have that  $Y$  is a weak Del Pezzo surface of one of the following types:

$(d, c)$	$2(d - c)$	$max$
$(1, 0), (2, 1),$	2	$\infty$
$(2, 0), (3, 1), (4, 2),$	4	10
$(4, 0), (6, 2), (7, 3), (8, 4)$	8	2

where the  $max$  column denotes the maximal number of complex families of circles. Note that  $2(d - c)$  is the Möbius degree of  $Y$ .

**Proof:** Let  $Z \subset \mathbf{P}^3$  be the projected model of  $Y$ . Let  $row$  be the row number of the table in Theorem 31 corresponding to  $Y$ . Let  $P(r) = \mathbf{Z}\langle H, F \rangle$  be the standard geometrically ruled basis.

*Claim 1:* We have that  $row \neq 8$  (cubic Hirzebruch surface).

Suppose by contradiction that  $row = 8$ . It follows that  $Y$  is a Hirzebruch surface with  $D = H + F$ . We have that  $H$  is the unique two dimensional class in which a conical family is contained. If cyclicity is  $c = 1$  then the class of the absolute conic must be  $H$ . It follows that the divisor class of a circle intersects  $H$  in two points. From  $H^2 = 1$  it follows that  $c \geq 2$  such that the degree of  $Z$  is three. ⚡

Let  $MTD(\mu, f, \beta)$  be the Möbius transformation diagram. Let  $M$  be the Möbius model of  $Y$ . Let  $p_a(D)$  be the arithmetic genus of  $D$  (this is equal to the geometric genus of a plane section of  $Z$ ).

*Claim 2:* If  $(d, c) = (4, 2)$  then  $p_a(D) = 1$ .

From Proposition 29 it follows that there exists a Möbius transformation  $\mu$  such that  $Z' = \mu(Z)$  is a celestial of type  $(3, 1)$ . This surface is obtained by projecting  $M$  from a simple point. The hyperplane sections of  $Z'$  pull back along this projection  $M \xrightarrow{f} \mathbf{P}^3$  to hyperplane sections with a simple point at the center of projection. Since the geometric genus is a birational invariant it follows that the geometric genus of plane sections of  $Z$  are equal to the geometric genus of plane sections of  $Z'$ . From Theorem 31 and claim 1) it follows that  $Z'$  is a weak Del Pezzo surface of degree three. The geometric genus of hyperplane sections of the anti-canonical model of a Del Pezzo surface is one, thus this claim follows.

*Claim 3:* We have that  $row \neq 2$  (Veronese surface) and  $row \neq 7$  (quartic Hirzebruch surface).

Suppose by contradiction that  $row = 2$  or  $row = 7$ . From  $2c \leq d$  It follows that  $Z$  is of type  $(4, 1)$  or  $(4, 2)$ . From the genus formula it follows that  $p_a(D) = \frac{1}{2}D(D + K) + 1 = 0$ . We have that  $H$  is the unique two dimensional class in which a conical family is contained. If cyclicity is  $c = 1$  then the class of the absolute conic must be  $H$ . It follows that the

divisor class of a circle intersects  $H$  in two points. From  $H^2 \in \{0, 1\}$  it follows that  $c = 2$ . From claim 2) it follows that  $p_a(D) = 1$ . ⚡

*Claim 4:* We have that  $Y$  is a weak Del Pezzo surface. This claim follows from claim 1) and claim 3).

*Claim 5:* We have that  $d - c \leq 4$  and  $2c \leq d \leq 8$ .

From Proposition 28 it follows that the Möbius model of  $Y$  is a multiple conical surface of degree  $2(d - c)$ . From Theorem 31 it follows that a multiple conical surface is of degree at most 8. From Bezout's theorem it follows that  $2c \leq d$ .

*Claim 6:* We have that  $(d, c)$  is not equal to  $(4, 1)$  or  $(5, 1)$ .

Assume by contradiction that  $(d, c)$  equals  $(4, 1)$ . From Theorem 31 it follows that a family of conics corresponds to either one of the following divisor classes  $H - Q_1, \dots, H - Q_5$  and  $2H - Q_1 - Q_2 - Q_3 - Q_4, \dots, 2H - Q_2 - Q_3 - Q_4 - Q_5$  with respect to the standard Del Pezzo basis. From  $c = 1$  it follows that without loss of generality we may assume that  $H - Q_5$  is the class of the absolute conic. The only family of conics with intersection product 2 with the absolute conic is  $2H - Q_1 - Q_2 - Q_3 - Q_4$ . It follows that  $Z$  has only one family of circles. ⚡ In a similar way it follows that  $Z$  has no family of circles if  $(d, c)$  equals  $(5, 1)$ .

*Claim 7:* We have that  $(d, c)$  is not equal to  $(3, 0)$ .

Assume by contradiction that  $(d, c)$  equals  $(3, 0)$ . The absolute conic intersects  $Z$  in a finite number of points. It follows that the families of circles of  $Z$  have at least two base points on the absolute conic. From Proposition 28 and Theorem 31 it follows that the Möbius model of  $Y$  is a weak Del Pezzo surface of degree 6. From claim 6) it follows that  $Y$  is not Möbius equivalent to a celestial of type  $(4, 1)$ . It follows that the Möbius model of  $Y$  does not have isolated double points. The isolated double points of  $Z$  must be the projection of two lines in the Möbius model, which intersect in an isolated triple point. ⚡

*Claim 8:* We have that  $d - c \neq 3$ .

Suppose by contradiction that  $d - c = 3$ . From claim 5), claim 6) and claim 7) it follows that  $Z$  is of type  $(5, 2)$  or  $(6, 3)$ . It follows that  $M$  is non-singular. Up to Möbius equivalence we may assume that  $Z$  is of type  $(6, 3)$ . From Theorem 31 it follows that  $Z$  and  $M$  are projections of an anti-canonical model of a weak Del Pezzo surface of degree six. It follows that the geometric genus of a generic hyperplane section  $S$  of  $Z$  is 1. From the genus formula for plane curves it follows that the delta invariants of  $S$ , at the two intersections with the triple absolute conic, add up to 9. ⚡

*Claim 9:* We have that **a**).

From claim 5) and Proposition 29 it follows that up to Möbius equivalence  $(d, c)$  equals either  $(2, 1)$ ,  $(4, 2)$ ,  $(6, 3)$  or  $(8, 4)$ . The maximal number of families circles is equal to the maximal families of conics and follows from Lubbes [2012b]. There are examples of celestials of type  $(2, 1)$ ,  $(2, 0)$ ,  $(4, 0)$  and  $(6, 2)$ . This claim follows from claim 8). ☹

**Remark 33. (celestial in higher dimensional space)** From Schicho's multiple conical surface classification we know that a celestial can be embedded in at most  $\mathbf{P}^8$ , such that the celestial is not contained in a linear subspace. In particular we have that a celestial is either of the following

- a Hirzebruch surface of degree  $3 \leq d \leq 4$  in  $\mathbf{P}^n$  for  $4 \leq n \leq 5$ , or
- a weak Del Pezzo surface of degree  $3 \leq d \leq 8$  in  $\mathbf{P}^n$  for  $3 \leq n \leq d$ .

Note that according to our definition, the plane  $\mathbf{P}^2$ , a quadric in  $\mathbf{P}^3$ , and a Veronese surface in  $\mathbf{P}^5$  are also weak Del Pezzo surfaces. See Definition 10 and Example 9 for the plane and Veronese surface.

The Möbius models of celestials in  $\mathbf{P}^3$  are again celestials in  $\mathbf{P}^4$  of degree 2, 4 or 8. The classification of celestials in  $\mathbf{P}^n$  for  $n > 3$  is still open.

## 8 Classification of celestials of Möbius degree 2

From Theorem 32 it follows that celestials of Möbius degree two are of type  $(1, 0)$  or  $(2, 1)$ . These are respectively the plane and the sphere. Note that the sphere is Möbius equivalent to the plane via stereographic projection. The family of circles going through the point of projection on the sphere is sent to a family of lines.

## 9 Classification of celestials of Möbius degree 4

**Remark 34. (summary and additional observations)** From Theorem 32 it follows that a real celestial  $Z$  of Möbius degree 4 is a weak Del Pezzo surface of type either  $(2, 0)$ ,  $(3, 1)$  or  $(4, 2)$ . Recall that  $Z$  in 3-space is the projection of the Möbius model  $M$  in the 3-sphere in projective 4-space. It follows that  $Z$  is Möbius equivalent to a (real) celestial of type  $(2, 0)$  if and only if  $M$  has a (real) double point.

A celestial of type  $(4, 2)$  is compact, since the plane at infinity is intersected in the double absolute conic, and thus in no real points. From a similar argument it follows that celestials of type  $(3, 1)$  are not compact. A real surface of type  $(3, 1)$  intersects the plane at infinity in the real absolute conic. The remaining line component in the plane at infinity section, must be real.

From Proposition 18 we obtain the classification of the isolated double points and the real structures of  $M$ . In other words, Proposition 18 classifies  $M$  up to Cremona equivalence. The classification of  $M$  up to (real) Cremona equivalence is also the classification of  $M$  up to (real) diffeomorphism since  $M$  is the anti-canonical model of a weak Del Pezzo surface.



For a given real Cremona equivalence class of a degree four weak Del Pezzo surface, it is a priori not clear whether there exists a representative with an anti-canonical model contained in the 3-sphere. We will show that this is the case for all such equivalence classes. Note that if  $M$  is contained in the 3-sphere then families of conics, are families of circles.

The first proposition in this section will state the classification of real celestials of type  $(2, 0)$  up to Euclidean equivalence. For each normal form for Euclidean equivalence classes of quadric surfaces, we consider the intersection with the absolute conic. There are 4 intersections, when counted with multiplicity. Let  $L$  be a real line through two conjugate intersection points. The sections by planes through  $L$  defines a family of circles or lines. Conversely, every family of circles is defined in this way. The reason is that circles are defined as conics which intersect the absolute conic in two different points.

It can happen that  $M$  contains a family of circles which has a real double point as base point. Note that this is completely determined by the real enhanced Picard group. We consider the class  $G$  of a family of conics in the real irreducible Del Pezzo two-set. We can define  $G$  as the unprojected class of a generic circle in the family. The unprojected class  $P$  of the double point is in the effective Del Pezzo zero-set. If  $GP \neq 0$  then the family of circles has a base point at the double point. Since a celestial of type  $(2, 0)$  is the projection of  $M$  from this double point it follows that the circles of  $G$  are projected to lines. In Proposition 37 the unprojected classes of families of circles and singularities are stated for each Cremona equivalence class.

For each Euclidean equivalence class of real celestials of type  $(2, 0)$  we consider its Möbius model and count the number of complex and real singularities. We count the number of families. We also count the number of pairs of families with intersection product two. We cross reference with the classification of real degree four weak Del Pezzo surfaces, and find that each entry in Table 21 with at least two families of real conics, and no real lines is reached. In combination with Theorem 32 this proves that a real celestial has at most 6 families.

The elliptic hyperboloid of two sheets (or EH2 for short) and the ellipsoid (or E for short) are not Euclidean equivalent, but their Möbius models turn out to be real Cremona equivalent and thus diffeomorphic. Although EH2 has two real components, its Möbius model has one component which is pinched at a double point. Indeed from Proposition 18 it follows that the only Cremona equivalence class of Möbius models with two real components is non-singular with real structure  $RI = 15$ .

From classically known examples of real celestials with degree 4 Möbius models without real singularities, we conclude the existence of the remaining entries in Table 21: Blum cyclide, sphere cyclide, cyclide with 2 components, Perseus cyclide and Dupin cyclide. The Blum cyclide has 6 families of circles. From Proposition 37 it follows that there are three pairs of co-spherical families. In other words, there are three pairs of families of circles such that a generic circle in one family intersects a circle in the other family in two points. The torus is real Cremona equivalent to a Dupin cyclide. In this case there are 4 families

of circles, with two of them co-spherical: the families of Villarceau circles. In Example 43 the torus is analyzed in more detail.

**Proposition 35. (classification of real celestials of type (2,0))**

Let  $Y = (X, D, V^3, \sigma)$  be a real celestial of type (2, 0). Let  $Z \subset \mathbf{P}^3$  be the projected model of  $Y$ .

a) Up to Euclidean equivalence  $Y$  is in the following table:

$Z \subset \mathbf{P}^3$	$\#C$	$\#R$	$\#F$	$\#P$	Description
$\alpha_0 x^2 + \alpha_1 y^2 - z^2 - w^2 = 0$	1	1	4	1	EH1
$\alpha_0 x^2 + \alpha_1 y^2 - z^2 + w^2 = 0$	1	1	2	1	EH2
$\alpha_0 x^2 + \alpha_1 y^2 + z^2 - w^2 = 0$	1	1	2	1	E
$\alpha_0 x^2 + \alpha_1 y^2 - z^2 = 0$	2	2	3	1	EO
$\alpha_0 x^2 - \alpha_1 y^2 - zw = 0$	1	1	2	0	HP
$\alpha_0 x^2 + \alpha_1 y^2 - zw = 0$	1	1	2	1	EP
$\alpha_0 x^2 + \alpha_0 y^2 - z^2 - w^2 = 0$	3	1	3	0	CH1
$\alpha_0 x^2 + \alpha_1 y^2 - w^2 = 0$	1	1	3	1	EY
$\alpha_0 x^2 + \alpha_0 y^2 - z^2 = 0$	4	2	2	0	CO
$\alpha_0 x^2 + \alpha_0 y^2 - w^2 = 0$	3	1	2	0	CY

where

- $\alpha_0 \neq \alpha_1 \in \mathbf{R}_{>0}$ ,
- $M$  is the Möbius model of  $Y$ ,
- $\#C$  : number of singularities of  $M$  (over the complex numbers),
- $\#R$ : number of real of singularities of  $M$ ,
- $\#F$ : number of real families of Möbius circles of  $Z$ , and
- $\#P$ : number of pairs of families which intersect each other twice, and
- $E$  = elliptic/ellipsoid,  $C$  = circular,  $H$  = hyperbolic/hyperboloid,  $1$  = of one sheet,  $2$  = of two sheets,  $O$  = cone,  $Y$  = cylinder and  $P$ =parabolic/paraboloid (note that there is ambiguity in the notation of  $Y$ , however this should not lead to problems).

**Proof:**

*Claim 1:* We have the first column of a).

We go through the classification of quadric surfaces up to Euclidean equivalence. Each entry gives a family of surfaces (for example CH1 parametrized by  $(\alpha_0, \alpha_1)$ ). We intersect the surface with generic coefficients with the absolute conic (so there are at most 4

intersections). We distinguish between the circular case ( $\alpha_0 = \alpha_1$ ) and the elliptic case ( $\alpha_0 \neq \alpha_1$ ). We consider the real lines through two conjugate points. Families of circles are defined by the sections of planes through these real lines. We consider a family of lines also a family of circles. For example in the HP and PY case the family of planes through the real line connecting two conjugate intersections with the absolute conic defines a family of lines.

We will present a proof for the CH1 (circular hyperboloid of one sheet aka cooling tower). The other cases can be shown in the same way. The ellipsoid is treated in [Hilbert and Cohn-Vossen \[1952\]](#). Let  $\text{MTD}(\mu, f, \beta)$  be the Möbius transformation diagram. Let  $Z : F = x^2 + y^2 - z^2 - w^2 = 0 \subset \mathbf{P}^3$  be CH1 for  $\alpha_0 = 1$ .

*Claim 2:* We have that  $M : c^2 + d^2 - de = a^2 + b^2 + de - e^2 = 0 \subset S^3$ .

In order to compute  $M = f^{-1}(Z)$  we eliminate  $x, y, z, w$  from the ideal  $\langle F, a - 2wx, b - 2wy, c - 2wz, d - (x^2 + y^2 + z^2 - w^2), e - (x^2 + y^2 + z^2 + w^2) \rangle$ .

*Claim 3:* The Jacobian matrix of  $M$  is:


$$\begin{bmatrix} 0 & 0 & 2c & 2d - e & -d \\ 2a & 2b & 0 & e & d - 2e \end{bmatrix}$$

Left to the reader.

*Claim 4:* The singularities of the  $M$  are  $(0 : 0 : 0 : 1 : 1)$  and  $(1 : \pm i : 0 : 0 : 0)$ .

Left to the reader to check that these are all the points on  $M$  where the rank of the Jacobian matrix is less than two.

*Claim 5:* We have that **a)** for CH1.

We consider the intersection points with the absolute conic:  $V(\langle \alpha_0 x^2 + \alpha_0 y^2 - z^2 - w^2, w, x^2 + y^2 + z^2 \rangle)$ . We may assume  $\alpha_0 = 1$ . By inspection we find that the number of families of Möbius circles is three: two families of lines, and the horizontal hyperplane sections. From claim 3) it follows that this claim holds. 

### **Theorem 36. (classification of real celestials of Möbius degree 4)**

Let  $Y = (X, D, V^3, \sigma)$  be a real celestial of Möbius degree 4. Let  $Z \subset \mathbf{P}^3$  be the projected model of  $Y$ . Let  $M \subset S^3$  be the Möbius model of  $Y$ .

**a)** Up to real Cremona equivalence  $M$  is in the following table:

CI	RI	Dynkin	$\#\mathbf{R}$	$\#F$	$(2, 0)$	$(3, 1)$	$(4, 2)$
16	13	$\emptyset$	0	6	—		Blum cyclide
16	14	$\emptyset$	0	2	—		sphere cyclide
16	15	$\emptyset$	0	2	—		2 components
17	13	$A_1$	1	4	EH1		
17	14	$A_1$	1	2	E or EH2		
19	13	$2A_1$	2	3	EO		
19	13	$2A_1$	0	5	—		Perseus cyclide
20	13	$A_2$	1	2	HP	butterfly	
20	14	$A_2$	1	2	EP		
21	13	$3A_1$	1	3	CH1		
24	13	$A_3$	1	3	EY		
25	13	$4A_1$	2	2	CO		
25	13	$4A_1$	0	4	—		Dupin cyclide
30	13	$A_3 + 2A_1$	1	2	CY		

where

- CI is the index in Table 19,
- RI is the index in Table 20,
- Dynkin column denotes the Dynkin type of the singularities of M over the complex numbers,
- $\#\mathbf{R}$  denotes the number of real singularities of M,
- $\#F$  denotes the number of real families of Möbius circles of M,
- see Proposition 35 for EH1, EH2, E, EO, HP, EP, CH1, EY, CO and CY,
- $\text{MTD}(\mu, f, \beta)$  is the Möbius transformation diagram,
- $(2, 0)$  describes  $\mu(Z)$  such that  $\mu(Z)$  is of type  $(2, 0)$  or — if  $\mu(Z)$  is not real or does not exists, and
- similar for the  $(3, 1)$  and  $(4, 2)$  column.

b) If Y and Y' are Cremona equivalent and  $RI \neq 15$  then their Möbius models M and M' in  $S^3$  are diffeomorphic. If  $RI=15$  then there are two diffeomorphic equivalence classes for M depending on the relation of the two real disconnected components; either one component contains the other, or not.

**Proof:** The “Blum cyclide” is named by Daniel Dreibelbis on his web site. Families of non-singular cyclides with  $RI=14$  and  $RI=15$  can be found in Takeuchi [2000]. The “Perseus cyclide” is named in Blum [1980]. The “butterfly” appears in Schicho [2001] and named by Josef Schicho. We recall that “Dupin cyclides” are by definition Möbius equivalent to the torus. These are all special cases of “Darboux cyclides”.

*Claim 1:* We have that  $\varphi_D(X) \cong M$  is a weak Del Pezzo surface of degree 4.

From Theorem 32 it follows that  $Y$  is a Del Pezzo surface. The anti-canonical model  $\varphi_D(X)$  is in  $\mathbf{P}^4$ . It follows that this claim holds.

Let  $G_{\mathbf{R}}(Y)$  be the real irreducible Del Pezzo two-set.

*Claim 2:* Each class in  $G_{\mathbf{R}}(Y)$  defines a family of circles.

From Proposition 17 it follows that  $\#G_{\mathbf{R}}(Y)$  is equal to the number of real families of conics of  $\varphi_D(X)$ . From claim 1) it follows that each family of real conics defines a family of circles.

*Claim 3:* We have that  $(CI, RI)$  is in

CI	RI	Dynkin	$\#C$	$\#R$	$\#F$	$\#P$
16	13	$\emptyset$	0	0	6	3
16	14	$\emptyset$	0	0	2	1
16	15	$\emptyset$	0	0	2	1
17	13	$A_1$	1	1	4	1
17	14	$A_1$	1	1	2	1
19	13	$2A_1$	2	2	3	1
19	13	$2A_1$	2	0	5	2
20	13	$A_2$	1	1	2	0
20	14	$A_2$	1	1	2	1
21	13	$3A_1$	3	1	3	0
24	13	$A_3$	1	1	3	1
25	13	$4A_1$	4	2	2	0
25	13	$4A_1$	4	0	4	1
30	13	$A_3 + 2A_1$	3	1	2	0

where  $\#C$ ,  $\#R$ ,  $\#F$  and  $\#P$  are as in Proposition 35.

From claim 1) it follows that  $\varphi_D(X) \subset S^3$  and thus does not contain real lines. From Proposition 18 it follows that  $RI$  is in  $\{13, 14, 15\}$ . From claim 1) it follows that we have to consider only the cases in Table 21. From claim 2) it follows that  $\#G_{\mathbf{R}}(Y) \geq 2$ . From Table 21 it follows that these are the possible entries for the  $CI$  and  $RI$  column. From Proposition 17 it follows that  $\#C$  equals the number of summands in the Dynkin type. From Proposition 17 it follows that  $\#C$  equals the components of the intersection diagram of  $F(Y)$  which is fixed under the real structure  $\sigma$ . We consider the pairs  $(CI, RI)$  as in

the first two columns. We fix the real structure  $\sigma$  indexed by RI and consider all possible choices for  $F(Y)$  with Dynkin type indexed by CI such that  $\sigma(F(Y)) = F(Y)$  (see [Lubbes \[2012a\]](#)). It follows that this claim holds.

We present a short example to illustrate the proof of claim 3). We use the notation from Definition 14 and Definition 16. If  $(CI, RI) = (20, 13)$  then we have the following choices for  $F(Y)$  with Dynkin type  $A_2$  such that  $\sigma(F(Y)) = F(Y)$ :  $\{12, 23\}$ ,  $\{12, 1145\}$ ,  $\{13, 1145\}$  or  $\{1245, 23\}$ . For each of these choices for  $F(Y)$  we find that  $\#F = 2$ ,  $\#P = 0$  and  $\sigma$  fixes  $F(Y)$  element wise.

*Claim 4:* If  $(\#C, \#R, \#F, \#P) = (1, 1, 2, 1)$  then **a**).

From Proposition 35 it follows that  $M$  is Möbius equivalent to  $EH_2$ ,  $E$  or  $EP$ . We consider the equation for  $EH_2$ ,  $E$  and  $EP$  in Proposition 35 with  $\alpha_0 = 1$  and  $\alpha_1 = 2$ . We apply a Möbius transform to a celestial of type  $(3, 1)$  and find that the Milnor number of the singularity is 1 for  $E$  and  $EH_2$ , and 2 for  $EP$ . (see [Greuel and Pfister \[2008\]](#), appendix A, section 9, example 4, page 528 for computing the Milnor number). The Milnor number of an  $A_n$  singularity is  $n$ . The Milnor number is an analytic invariant and thus it follows that the singularity is of type  $A_n$ .


*Claim 5:* If  $\#R \neq 0$  then **a**).

We have that  $M$  has a real singularity. We choose  $\beta$  such that  $f$  is the projection from this real singularity. We have that  $Z$  is Möbius equivalent to a celestial of type  $(2, 0)$ . By cross referencing with the table in Proposition 35 we find the possible entries in claim 3). From claim 4) it follows that this claim holds.

*Claim 6:* If  $\#R = 0$  then **a**).

If  $\#R = 0$  then  $Z$  is not Möbius equivalent to a real celestial of type  $(2, 0)$ . From claim 3) it follows the possible singularity types and real structures. The existence of such celestials follows from known examples (see §10).

*Claim 7:* We have that **b**).

From claim 1) it follows that  $Y$  and  $Y'$  are two anti-canonical models of weak Del Pezzo surfaces of degree 4. We have that their enhanced Picard groups are isomorphic, and thus their singularities have the same Dynkin type. Locally, singularities of the same Dynkin type are diffeomorphic (see [Arnol'd et al. \[1985\]](#)). If there is one real component, then these local diffeomorphisms can be extended to a global diffeomorphism. If there are two real components then these must be disconnected, otherwise the real singularity decreases the geometric genus of a generic hyperplane section. Up to diffeomorphism we only have to distinguish between two concentric spheres or two exclusive spheres (see also [Takeuchi \[1987\]](#)). 

See the analysis of the Dupin cyclide in Example 43 for an application of the following proposition.

**Proposition 37. (unprojected classes of families of circles and singularities)**

Let  $Y = (X, D, V^3, \sigma)$  be a real celestial of Möbius degree 4.

**a)** Up to real Cremona equivalence we have the following table of unprojected classes of families of circles and singularities of  $Y$ :

CI	RI	Dynkin type	$F(Y)$	$\sigma(F(Y))$	$G_{\mathbf{R}}(Y)$
16	13	$\emptyset$	$\emptyset$	$\emptyset$	$a1, a2, a3, b1, b2, b3$
16	14	$\emptyset$	$\emptyset$	$\emptyset$	$a1, b1$
16	15	$\emptyset$	$\emptyset$	$\emptyset$	$a1, b1$
17	13	$A1$	12	12	$a1, a3, b3, b2$
17	14	$A1$	1145	1145	$a1, b1$
19	13	$2A1$	12, 1345	12, 1345	$a1, a3, b3$
19	13	$2A1$	14, 1235	1235, 14	$a1, a2, a3, b2, b3$
20	13	$A2$	12, 23	12, 23	$a1, b3$
20	14	$A2$	24, 1125	1125, 24	$a1, b1$
21	13	$3A1$	12, 1125, 34	12, 34, 1125	$a1, a3, b2$
24	13	$A3$	12, 1135, 24	12, 24, 1135	$a1, a3, b3$
25	13	$4A1$	12, 1345, 1125, 34	12, 1345, 34, 1125	$a1, a3$
25	13	$4A1$	14, 1134, 25, 1235	1235, 25, 1134, 14	$a1, a2, a3, b3$
30	13	$A3 + 2A1$	12, 35, 1135, 24, 1124	12, 1124, 24, 1135, 35	$a1, a3$

where

- $CI$ ,  $RI$  and Dynkin type are as in Theorem 36,
- $F(Y)$  is the effective Del Pezzo zero-set (see Definition 14 for short notation of its elements),
- $\sigma(F(Y))$  is  $F(Y)$  permuted by  $\sigma$  (so the elements of both columns are ordered), and
- $G_{\mathbf{R}}(Y)$  is the real irreducible Del Pezzo two-set (see Definition 16 for short notation of its elements).

**Proof:**

*Claim:* We have that **a)**.

We consider the  $CI$ ,  $RI$  and  $\#F$  as in Theorem 36. We fix the real structure  $RI$  as in Table 20. We consider all possible  $F(Y)$  such that its Dynkin type corresponds to the Dynkin type indexed by  $CI$ . For each  $F(Y)$  we obtain  $G_{\mathbf{R}}(Y)$ . We consider different  $F(Y)$ , such that  $G_{\mathbf{R}}(Y)$  has the same intersection diagram, equivalent. ☹️

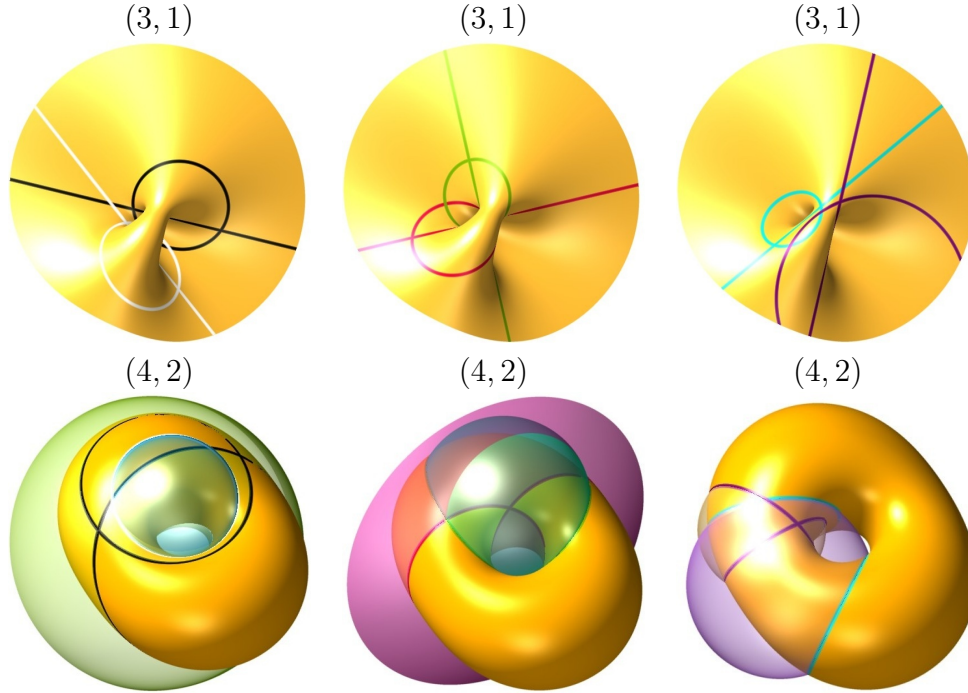
## 10 Examples of celestials of Möbius degree 4

In the following definition we introduce some notation, which enables us to reconstruct the examples of celestials in this paper.

**Definition 38. (notation)** Let  $\text{MTD}(\mu, f, \beta)$  be the Möbius transformation diagram. Let  $\beta_0 : S^3 \rightarrow S^3$ ,  $(a : b : c : d : e) \mapsto (a : b : c : d : -e)$ . Let  $\beta_1 : S^3 \rightarrow S^3$ ,  $(a : b : c : d : e) \mapsto (a : b : d : c : e)$ . Let  $\beta_2 : S^3 \rightarrow S^3$ ,  $(a : b : c : d : e) \mapsto (a : b : c : -d : e)$ . Let  $t : \mathbf{P}^3 \rightarrow \mathbf{P}^3$ ,  $(x : y : z : w) \mapsto (x - t_0 w : y - t_1 w : z - t_2 w : w)$  be a translation. We define the Möbius transformations  $\mathbf{P}^3 \xrightarrow{\mu_i} \mathbf{P}^3$  which are used for the examples below:  $\mu_0 := t \circ f \circ \beta_2 \circ f^{-1}$  with  $(t_0, t_1, t_2) = (-1, -1, -1)$ ,  $\mu_1 := t \circ f \circ \beta_1 \circ f^{-1}$  with  $(t_0, t_1, t_2) = (0, 1, 0)$ ,  $\mu_2 := t \circ f \circ \beta_1 \circ f^{-1}$  with  $(t_0, t_1, t_2) = (1, 1, 0)$ ,  $\mu_3 := t \circ f \circ \beta_0 \circ f^{-1}$  with  $(t_0, t_1, t_2) = (0, 1, 1)$ ,  $\mu_4 := t \circ f \circ \beta_0 \circ f^{-1}$  with  $(t_0, t_1, t_2) = (0, 1, 0)$ ,  $\mu_5 := f \circ \beta_1 \circ f^{-1}$ ,  $\mu_6 := f \circ \beta_0 \circ f^{-1}$ ,  $\mu_7 := f \circ \beta_0 \circ f^{-1} \circ t$  with  $(t_0, t_1, t_2) = (1, 0, 0)$ ,  $\mu_8 := f \circ \beta_1 \circ f^{-1} \circ t$  with  $(t_0, t_1, t_2) = (1, 0, 0)$ , and  $\mu_9 := f \circ \beta_0 \circ f^{-1} \circ t$  with  $(t_0, t_1, t_2) = (1, 1, 0)$ . Note that it would also be possible to express the  $\mu_i$  as  $f \circ \beta \circ f^{-1}$  for some  $\beta$ . We will represent the families of circles in celestials  $Z \subset \mathbf{P}^3$  of type  $(3, 1)$  as planes  $F_i$  which go through a real line. These planes are parametrized by  $t \in \mathbf{R}$ . The Möbius transform to a celestial of type  $(4, 2)$  sends the planes to spheres. Note that the spheres go to a fixed circle, namely the Möbius transform of the line. Thus each sphere intersects a celestial in two circles. One of these circles might be in another family. The reason is that a special element in a family of circles might degenerate as two lines. Two families of circles can contain the same line component.

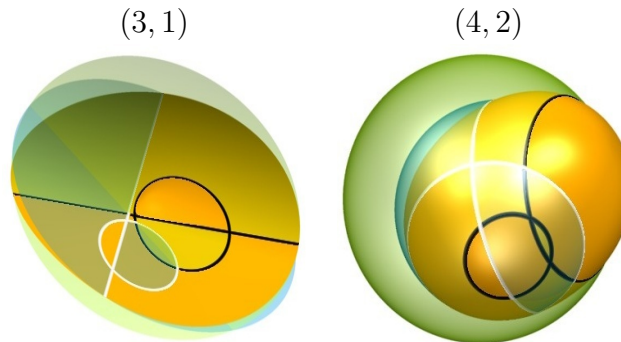


**Example 39. (Blum cyclide)** The Blum cyclide has no isolated singularities and 6 families.



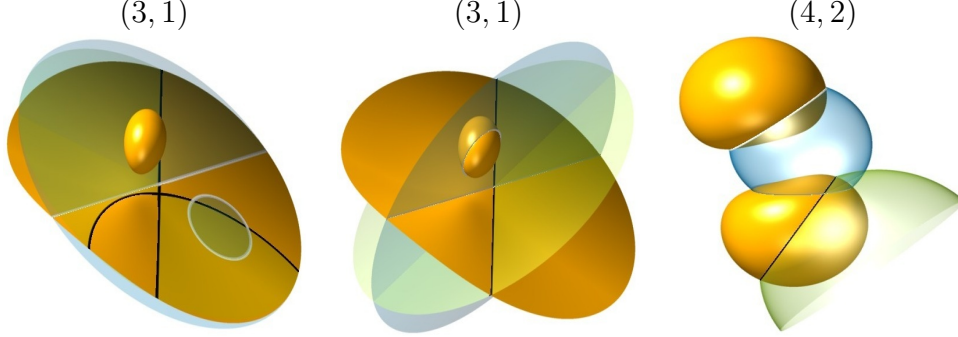
$Z : 2(y + 3z)(x^2 + y^2 + z^2) + w(4(x + y + z + 1)^2 + 2y^2 + 6z^2) = 0$ ,  $F_1 : (y + 3z + 1) + t(x - (-\sqrt{6} + 4)z) = 0$ ,  $F_2 : (y + 3z + 1) + t(x - (\sqrt{6} + 4)z) = 0$ ,  $F_3 : (y + 3z + 2) + t(2x + 1) = 0$ ,  $F_4 : (y + 3z + 2) + t(2z + 1) = 0$ ,  $F_5 : (y + 3z + 3) + t(-4 + \sqrt{6} - x + (-4 + \sqrt{6})z) = 0$ ,  $F_6 : (y + 3z + 3) + t(4 + \sqrt{6} + x + (4 + \sqrt{6})z) = 0$ ,  $t \in \mathbf{R}$ , the Möbius transform from (3, 1) to (4, 2) is defined by  $\mu_0$ . See Definition 38 for the notation.

**Example 40. (sphere cyclide)** The sphere cyclide has no isolated singularities  $\emptyset$  and 2 families.



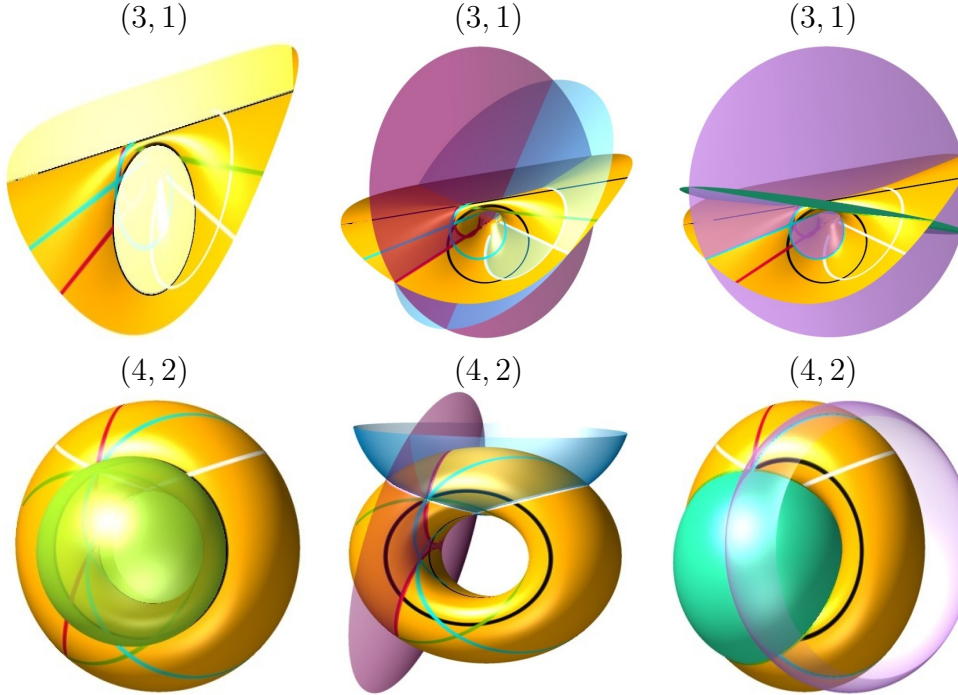
$Z : 2(y - \frac{3}{2}z)(x^2 + y^2 + z^2) + 2w(x^2 - \frac{3}{2}y^2 + 2yz + yw + z^2 - \frac{3}{2}zw) = 0$ ,  $F_1 : (2y - 3z + w) + t(x + (\sqrt{2})z + (-\sqrt{2})w) = 0$ ,  $F_2 : (2y - 3z + w) + t(x + (-\sqrt{2})z + (\sqrt{2})w) = 0$ ,  $t \in \mathbf{R}$ , the Möbius transform from (3, 1) to (4, 2) is defined by  $\mu_1$ . See Definition 38 for the notation.

**Example 41. (cyclide with two components)** The cyclide with two components has no isolated singularities and 2 families.



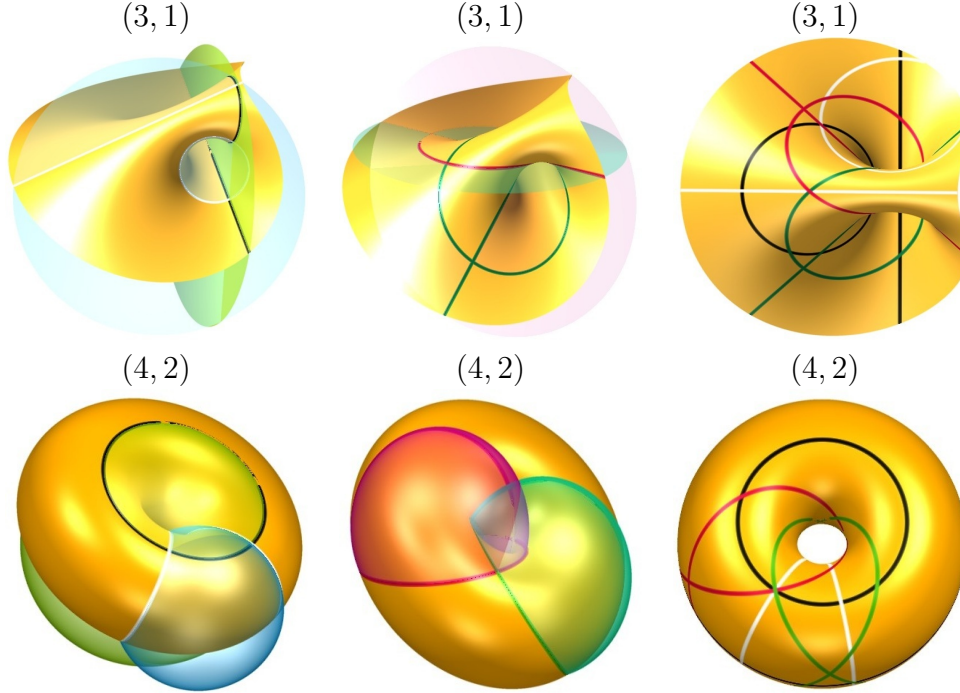
$Z : 6(x - \frac{5}{6}y + z)(x^2 + y^2 + z^2) - 8w(x^2 - xy + \frac{5}{2}xz - \frac{3}{4}xw - \frac{3}{8}y^2 - \frac{1}{4}yz + \frac{5}{8}yw + z^2 - \frac{3}{4}zw) = 0$ ,  
 $F_1 : (x - \frac{5}{6}y + z) + t(y + (\frac{12}{37}\sqrt{3} - \frac{42}{37})z) = 0$ ,  $F_2 : (x - \frac{5}{6}y + z) + t(y + (-\frac{12}{37}\sqrt{3} - \frac{42}{37})z) = 0$ ,  
 $t \in \mathbf{R}$ , the Möbius transform from (3, 1) to (4, 2) is defined by  $\mu_2$ . See Definition 38 for the notation.

**Example 42. (Perseus cyclide)** The Perseus cyclide has two isolated singularities of Dynkin type  $2A_1$  and 5 families.



$Z : 4(x + y + z + 1)^2 + 5(x^2 + y^2 + z^2)z + 5z^2 = 0$ ,  $F_1 : (z) + t(x + y + w) = 0$ ,  $F_2 : (z + w) + t(x + (-4 + \sqrt{15})y) = 0$ ,  $F_3 : (z + w) + t(-x + (4 + \sqrt{15})y) = 0$ ,  $F_4 : (5z + 2w) + t(-6w + 2\sqrt{3}w - 5x + (-10 + 5\sqrt{3})y) = 0$ ,  $F_5 : (5z + 2w) + t(6w + 2\sqrt{3}w + 5x + (10 + 5\sqrt{3})y) = 0$ ,  
 $t \in \mathbf{R}$ , the Möbius transform from (3, 1) to (4, 2) is defined by  $\mu_0$ . See Definition 38 for the notation.

**Example 43. (Dupin cyclide)** The torus is a special example of a Dupin cyclide. The Dupin cyclide has 4 singularities of Dynkin type  $A_1$  and 4 families.



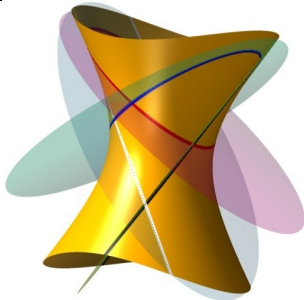
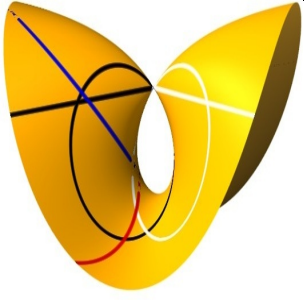
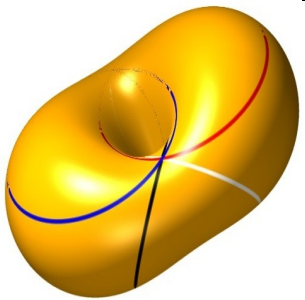
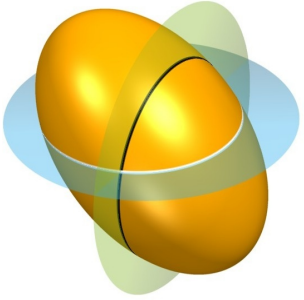
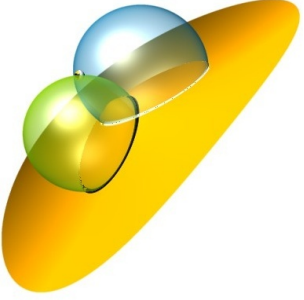
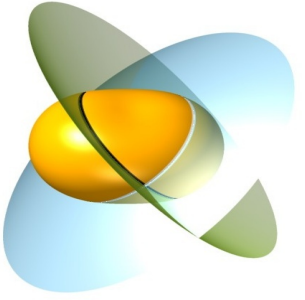
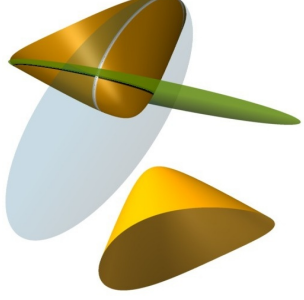
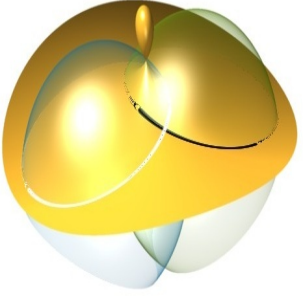
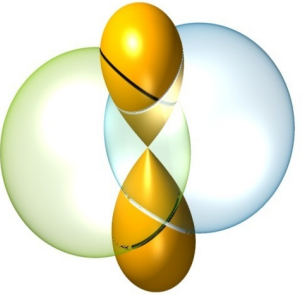
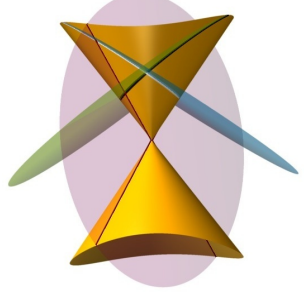
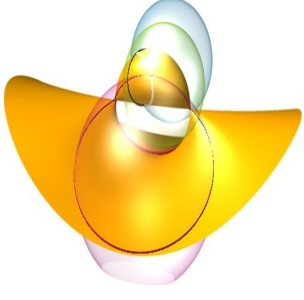
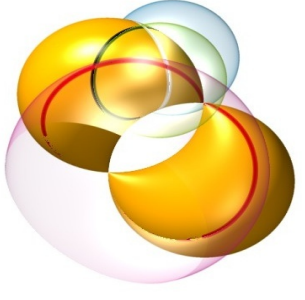
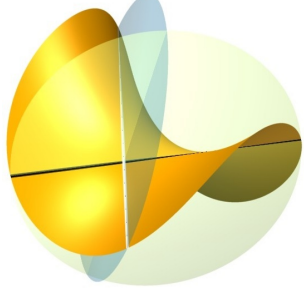
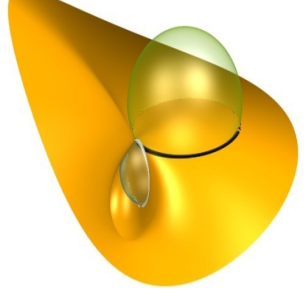
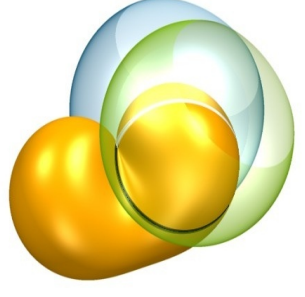
$Z : 2z(x^2 + y^2 + z^2) + w((x + y + z + w)^2 + 2z^2) = 0$ ,  $F_1 : z + t(y + x + w) = 0$ ,  $F_2 : (2z + 2w) + t(x - y) = 0$ ,  $F_3 : (2z + w) + t(y - z) = 0$ ,  $F_4 : (2z + w) + t(x - z) = 0$ ,  $t \in \mathbf{R}$ , the Möbius transform from  $(3, 1)$  to  $(4, 2)$  is defined by  $\mu_0$ . See Definition 38 for the notation. From Proposition 37 it follows that the classes of the families of circles are  $a_1, a_2, a_3$  and  $b_3$ . From Definition 16 it follows that  $a_3 \cdot b_3 = 2$  and the remaining pairwise intersections are 1. It follows  $a_3$  and  $b_3$  are the families of circles defined in the second column above. On the torus these circles are called the *Villarceau circles*.

The two pairs of conjugate singular points have the unprojected classes  $(14, 1235)$  and  $(25, 1134)$ . See Definition 14 for the notation. We see that  $a_1 \cdot 14 = a_1 \cdot 1235 = 1$  and  $a_2 \cdot 25 = a_2 \cdot 1134 = 1$ . Thus  $a_1$  has two base points 14 and 1235. Consequently we find that  $a_1$  is defined by hyperplane sections through the line connecting the points with unprojected classes 14 and 1235. Similar for  $a_2$ . We have that  $a_1$  and  $a_2$  are the families in the first column.


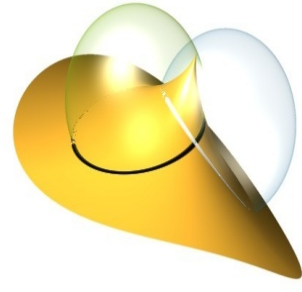
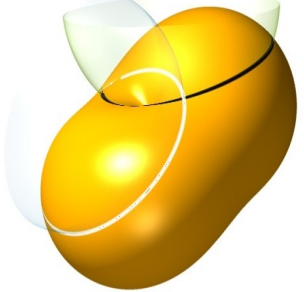
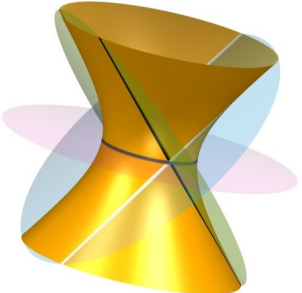
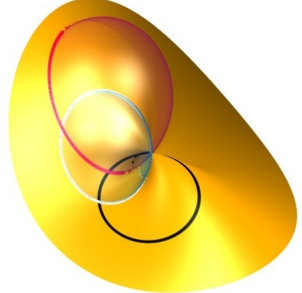
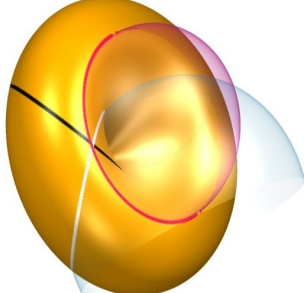
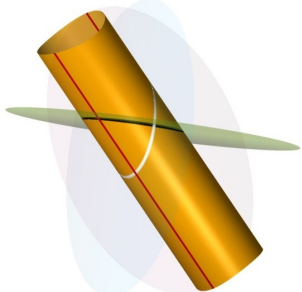
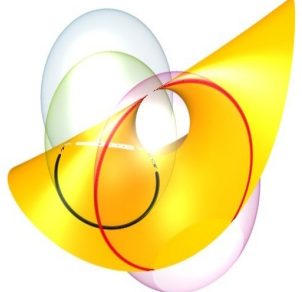
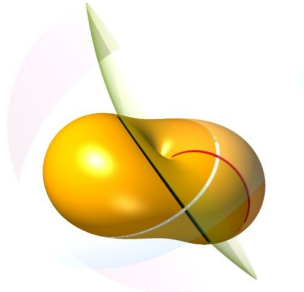
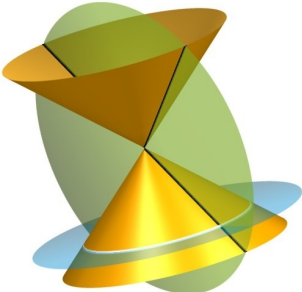
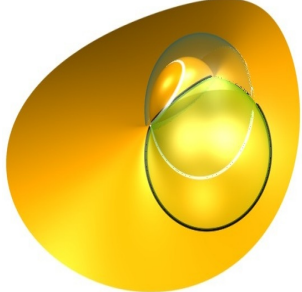
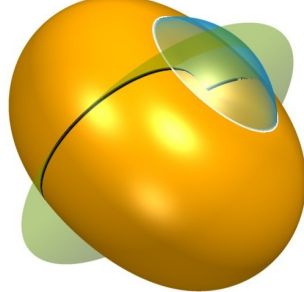
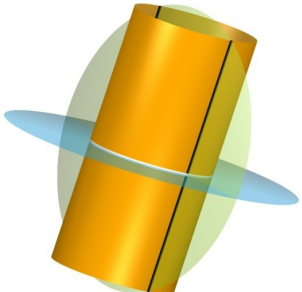
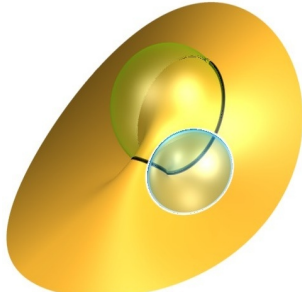
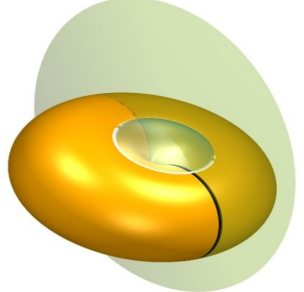
From the equation of the torus we find that two double points of the Möbius model are projected to the absolute double conic. Let us assume without loss of generality that these are 14 and 1235 and defined by  $(\pm i : 1 : 0 : 0)$ . Then  $a_1$  is the family corresponding to the horizontal hyperplane sections of the torus and  $a_2$  is the family of rotating circles with base points  $(0 : 0 : \pm i : 1)$ .

**Example 44. (celestials with real isolated singularities)**

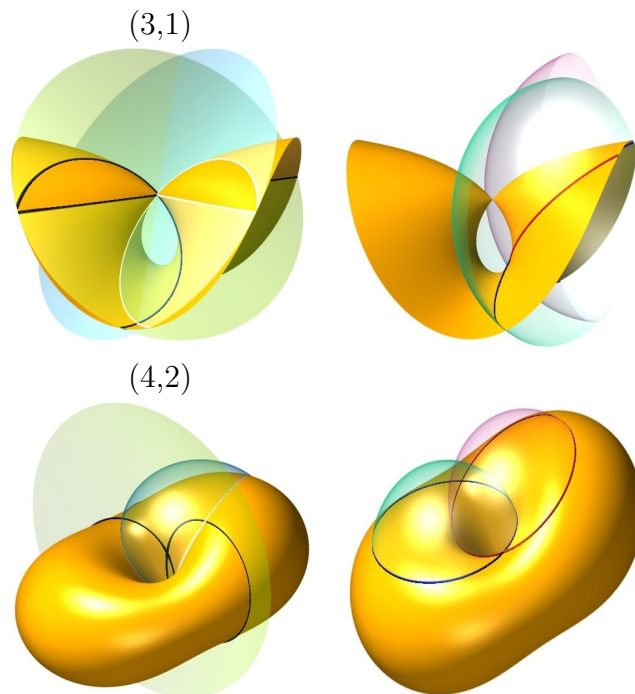
Celestials with real isolated singularities are Möbius equivalent to celestials of type  $(2, 0)$ . We use the notation from Theorem 36 and Definition 38.

(CI,RI,.)	(2, 0)	(3, 1)	(4, 2)
(17,13,A1)			
(17,14,A1)			
(17,14,A1)			
(19,13,2A1)			
(20,13,A2)			

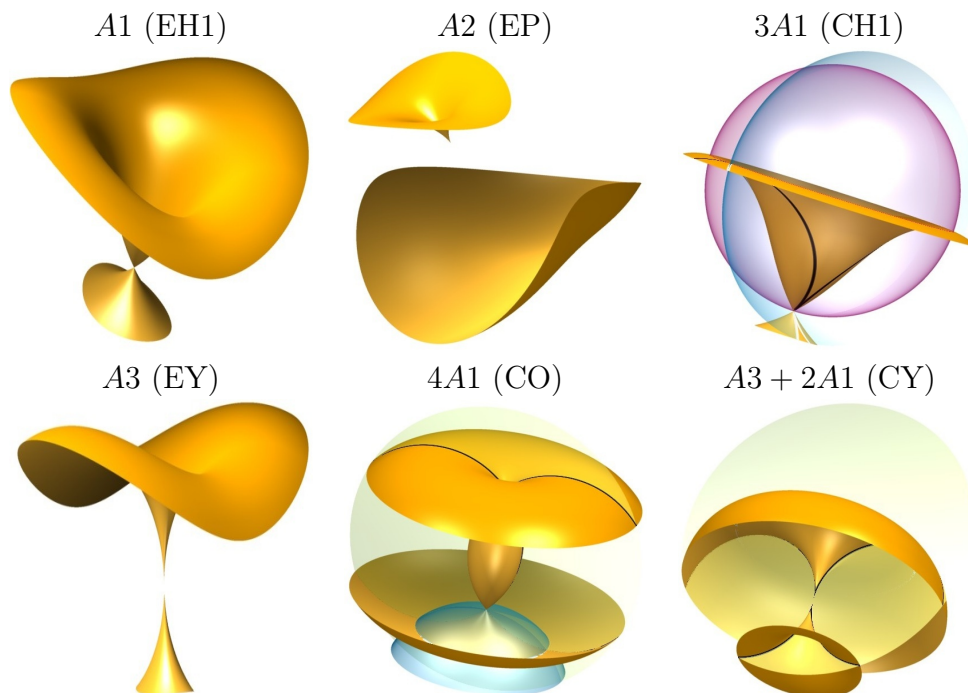


(20,14,A2)			
(21,13,3A1)			
(24,13, A3)			
(25, 13,4A1)			
(30, 13, A3 + 2A1)			

Here is the EH1 from the first row with the families in separate images.



Below, spherical clippings of the  $(4, 2)$  transforms. We can see the singularities inside the surface.



The equations for  $Z$  of type  $(2,0)$  are as in Proposition 35 with  $\alpha_0 = 1$  and  $\alpha_1 = 2$ , except for the ellipsoid we have  $\alpha_0 = \alpha_1 + 1 = 3$ .

The first row depicts EH1 whose Möbius model has singularity configuration of Dynkin type  $A_1$ . Its Möbius transform to the depicted surface of type  $(3,1)$  is  $\mu_7$ . Its Möbius transform to the depicted surface of type  $(4,2)$  is  $\mu_5$ . We will denote this by the short notation  $(EH1, A_1, \mu_5, \mu_6)$ .

In this notation the remaining rows are respectively defined by:  $(E, A_1, \mu_5, \mu_6)$ ,  $(EH2, A_1, \mu_5, \mu_6)$ ,  $(EO, 2A_1, \mu_6, \mu_7)$ ,  $(HP, A_2, \mu_3, \mu_5)$ ,  $(EP, A_2, \mu_6, \mu_5)$ ,  $(CH1, 3A_1, \mu_4, \mu_5)$ ,  $(EY, A_3, \mu_7, \mu_1)$ ,  $(C0, 4A_1, \mu_3, \mu_5)$ , and  $(CY, A_3 + 2A_1, \mu_4, \mu_5)$ .

## 11 Classification of celestials of Möbius degree 8

**Remark 45. (summary)** From Theorem 32 we know that a celestial  $Y$  of Möbius degree 8 with projected model  $Z$  in 3-space is of type either  $(4,0)$ ,  $(6,2)$ ,  $(7,3)$  or  $(8,4)$  and has 2 families of circles. The normalization of the Möbius model  $M$  of  $Y$  is a non-singular degree eight Del Pezzo surface, the 2-uple embedding of a smooth quadric. The families of the lines on the quadric are sent to circles. We would like to know when  $Y$  is Möbius equivalent to a celestial of type  $(4,0)$ . For this purpose we would like to classify projections  $M$  of degree eight Del Pezzos in  $\mathbf{P}^8$  into the Möbius 3-sphere  $S^3$ .

In order to classify celestials of Möbius degree 8 in 3-space we first consider the unprojected classes of a generic circle in each family of circles and the unprojected class of the components of the plane at infinity section. This reveals some of the geometry of such celestials.

If  $Z$  is of type  $(6,2)$  the plane at infinity section consists of the double absolute conic and two lines (possibly a double line). These lines are components of degenerated conics in the families of circles. We have that  $Y$  is the projection from its Möbius model. The plane at infinity section pulls back along this projection to a degree 8 curve with a quartic point at the center of projection. The double point is blown up to the two lines in the plane at infinity.

If  $M$  contains a point of multiplicity 4 then  $Y$  is Möbius equivalent to a surface of type  $(4,0)$ . The absolute conic is not contained in the projected model  $Z$ . It follows that each family of circles must have two base points on the absolute conic. These base points must be singular points. By analyzing the unprojected class of the plane at infinity section we will see that up to Cremona equivalence there are 4 double points. We obtain an explicit description of the classes corresponding to the families of circles. We have that  $M$  has no isolated singularities. It follows that the 4 double points must come from 4 complex lines through the center of projection. These 4 complex lines lie in the pull back of the plane at infinity. The center of projection blows up to 4 lines in the plane at infinity. The latter 4 lines intersect in the 4 double points.

Before we analyze unprojected classes in the  $(6, 2)$  and  $(7, 3)$  case we first warm up with the  $(8, 4)$  and  $(7, 3)$  case. After that we partially analyze the singular locus of  $M$  in Proposition 52. We find that the singular locus of  $M$  is of at most degree eight, has no isolated double points, and there are no complex non-singular lines in  $M$ .

The real structure of  $Y$  is stated in Proposition 53. We delayed the analysis of the real structure as long as possible for more clarity from a logic point of view. However we note that some of the arguments could have been shortened if the real structure would have been used earlier. Theorem 54 extends Proposition 52 by also taking the real structure into account.

The real structure of a celestial of type  $(6, 2)$  is the identity. It follows that  $Z$  in this case is of degree 6, has 6 real lines and no isolated singularities. We know that  $M$  in the Möbius 3-sphere contains no real lines. Two lines come from the blow up of the center of projection from  $M$ . The remaining 4 lines come from 4 circles through a singular point of  $M$ . Note that through a generic point on  $M$  there go at most 2 circles.

If  $Y$  is real Möbius equivalent to a surface of type  $(4, 0)$ , then the singular locus of  $M$  consists of 4 conjugate double lines in  $M$  meeting in the center of projection, and either: (two double conics which meet at the center of projection) or (a real double conic going through 2 conjugate lines).

We start this section by characterizing classes of families of circles of  $Y$  by their intersection product with the absolute conic, and unprojected classes of singularities. Maybe you want to take a quick look to Definition 5 and Proposition 17 before proceeding with this section.

**Proposition 46. (properties of divisor classes of celestials of Möbius degree 8)**

Let  $Y = (X, D, V^3)$  be a celestial of Möbius degree 8. Let  $Z \subset \mathbf{P}^3$  be the projected model of  $Y$ .

a) We have that  $Y$  has at most two families of circles.

Let  $(W, F)$  be the strict unprojected class of the plane at infinity section of  $Z$ . Let  $(A, R)$  be the strict unprojected class of the absolute conic in  $Z$ . Let  $G(Y)$  be the irreducible Del Pezzo two-set.

b) If  $G \in G(Y)$  defines a family of circles then  $G(A + F) = 2$ .

**Proof:**

*Claim 1:* We have that a).

This claim follows from Theorem 32.

Let  $\varphi_D(X) \xrightarrow{\pi} Z$  be the projection which commutes with  $\varphi_{V^3}$ . Let  $A' \subset \varphi_D(X)$  be a quartic curve such that the divisor class of  $A'$  is  $A$ . Let  $\tilde{A} = \pi(A') \subset Z$ . Let  $G' \subset \varphi_D(X)$  be a generic conic such that the divisor class of  $G'$  is  $G$ . Let  $\tilde{G} = \pi(G') \subset Z$ .

*Claim 2:* We have that  $\#(\tilde{G} \cap \tilde{A}) \geq GA$  and  $0 \leq GA \leq 2$ .

We have that  $G'$  intersects  $A'$  in  $GA$  moving intersection points. The ‘moving’ happens



when letting  $G'$  move in the family defined by  $G$ . These points are projected to  $GA$  moving intersection points on  $\tilde{A}$ . From  $\tilde{G}$  and  $\tilde{A}$  not lying in the same plane it follows that  $GA \leq 2$ . From  $G$  having no fixed components it follows that  $GA \geq 0$ .

*Claim 3:* We have that  $GF \leq 2$ .

From Proposition 17 it follows that base points are determined by summands of  $F$ . It follows that  $F$  defines singular base points which lie in the plane at infinity. We have that  $G$  intersects the plane at infinity at most twice.

*Claim 4:* If  $\#(\tilde{G} \cap \tilde{A}) > GA$  then  $\tilde{G}$  intersects  $\tilde{A}$  in base points defined by  $F$ .

Non-generically it can happen that curves on  $Z$  intersect in the singular locus, but their unprojected classes do not intersect. A curve in a family generically intersects a non-moving point if and only if this is a base point.

*Claim 5:* We have that **b**).

From claim 2) and claim 4) it follows that  $GF + GA = \#(\tilde{G} \cap \tilde{A})$ . A generic circle in  $G$  intersects the absolute conic in two different points. ☕

**Proposition 47. (properties of celestials of type (8,4))**

Let  $Y = (X, D, V^3)$  be a celestial of type (8, 4). Let  $Z \subset \mathbf{P}^3$  be the projected model of  $Y$ . Let  $W'$  be the plane at infinity section of  $Z \subset \mathbf{P}^3$ .

**a)** We have that  $W'$  is the fourfold absolute conic.

Let  $A(Y) = (\text{Pic}(X), D, \cdot, h)$  be the enhanced Picard group. Let  $P(r) = \mathbf{Z}\langle H, F \rangle$  be the geometrically ruled basis.

**b)** We have that  $A(Y) \cong P(0)$ .

Let  $C$  be the unprojected class of a singular double conic in  $Z$ .

**c)** We have that  $C \in \{2H, 2F, H + F\}$ .

Let  $W$  be the unprojected class of  $W'$ .

**d)** We have that  $W = 2H + 2F$ .

Let  $F(Y)$  be the effective Del Pezzo zero-set. Let  $G(Y)$  be the irreducible Del Pezzo two-set. Let  $E(Y)$  be the Del Pezzo one-set. Let  $G_{\text{circles}}(Y) \subset G(Y)$  be the classes of families of circles.

**e)** We have that  $E(Y) = F(Y) = \emptyset$  and  $G(Y) = G_{\text{circles}}(Y) = \{H, F\}$ .

**Proof:**

*Claim 1:* We have that **a**).

Follows from the definitions.

*Claim 2:* We have that **b**) and **d**).

From Theorem 32 it follows that  $Y$  is a weak Del Pezzo surface. From Lubbes [2012b] and

Proposition 12 it follows that  $\#G(Y) \geq 2$  if and only if  $A(Y) \cong P(0)$  with  $D = 2H + 2F$ . We have that  $D = W$  is the class of the pull back of hyperplane sections.

Let  $C = c_0H - c_1F$ .

*Claim 3:* We have that **c**).

From  $CH > 0$  and  $CF > 0$  it follows that  $c_0, c_1 \in \mathbf{Z}_{\geq 0}$ . From  $DA = 4$  it follows that  $2c_0 + 2c_1 = 4$ . From these Diophantine equations it follows that this claim holds.

*Claim 4:* We have that **e**).

From Proposition 46 and claim 3) it follows that  $HA = FA = 2$  with  $DA = 8$ . This claim follows from claim 2). ☕

**Proposition 48. (properties of celestials of type (7,3))**

Let  $Y = (X, D, V^3)$  be a celestial of type (7, 3). Let  $Z \subset \mathbf{P}^3$  be the projected model of  $Y$ . Let  $W'$  be the plane at infinity section of  $Z \subset \mathbf{P}^3$ .

**a)** We have that  $W' = A' \cup L'$  where  $A'$  is the triple absolute conic, and  $L'$  is a line.

Let  $A(Y) = (\text{Pic}(X), D, \cdot, h)$  be the enhanced Picard group. Let  $B(2) = \mathbf{Z}\langle H, Q_1, Q_2 \rangle$  be the standard Del Pezzo basis. Let  $F(Y)$  be the effective Del Pezzo zero-set .

**b)** We have that  $A(Y) \cong B(2)$  and  $F(Y) = \emptyset$ .

Let  $(C, R)$  be the strict unprojected class of a singular double conic in  $Z$ .

**c)** We have that  $C \in \{ 2H - Q_1 - Q_2, 2(H - Q_1), 2(H - Q_2) \}$ .

Let  $A$  be the unprojected class of  $A'$ . Let  $B$  be the unprojected class of  $L'$ . Let  $W$  be the unprojected class of  $W'$ .

**d)** We have that  $W = A + B = 3H - Q_1 - Q_2$ .

Let  $G(Y)$  be the irreducible Del Pezzo two-set. Let  $E(Y)$  be the Del Pezzo one-set. Let  $G_{\text{circles}}(Y) \subset G(Y)$  be the classes of families of circles.

**e)** We have that

- $A = 2H, B = H - Q_1 - Q_2,$
- $E(Y) = \{ Q_1, Q_2, H - Q_1 - Q_2 \},$  and  $G(Y) = G_{\text{circles}}(Y) = \{ H - Q_1, H - Q_2 \}.$

**Proof:**

*Claim 1:* We have that **a**).

Left to the reader.

*Claim 3:* We have that **b**) and **d**).

From Theorem 32 it follows that  $Y$  is a weak Del Pezzo surface. From Proposition 12 it follows that  $A(Y) \cong B(2)$  with  $D = 3H - Q_1 - Q_2$ . We have that  $D = W$  is the class of

the pull back of hyperplane sections. From Lubbes [2012b] it follows that  $\#G(Y) \geq 2$  if and only if  $F(Y) = \emptyset$ .

*Claim 4:* We have that  $E(Y)$ ,  $G(Y)$  and  $G_{circles}(Y)$  as in **e**).

From Proposition 17 it follows that  $\#G_{circles}(Y) = 2$ . See Lubbes [2012a] for the remaining.

*Claim 5:* We have that  $CE \geq 0$  for all  $E \in E(Y)$ .

Assume by contradiction that  $CE < 0$  for some  $E \in E(Y)$ . We have that  $E$  is a fixed component of  $A$ . It follows that the absolute conic has a line as component. ⚡

Let  $C = c_0H - c_1Q_1 - c_2Q_2$ .

*Claim 6:* We have that **c**).

From claim 4) and claim 5) it follows that  $c_1, c_2 \in \mathbf{Z}_{\geq 0}$  and  $c_0 \geq c_1 + c_2$ . From  $DA = 4$  it follows that  $3c_0 = c_1 + c_2 + 4$ . From these Diophantine equations it follows that  $c_1 + c_2 \leq 2$  and  $c_0 \leq 2$ .

*Claim 7:* We have that **e**).

From Proposition 46 and claim 4) it follows that  $(H - Q_1)A = (H - Q_2)A = 2$  with  $DA = 6$ . From Proposition 17 it follows that  $B \in E(Y)$ . ☕

**Proposition 49. (properties of celestials of type (6,2))**

Let  $Y = (X, D, V^3)$  be a celestial of type (6, 2). Let  $Z \subset \mathbf{P}^3$  be the projected model of  $Y$ . Let  $W'$  be the plane at infinity section of  $Z \subset \mathbf{P}^3$ .

**a)** We have that

$$W' = A' \cup L'_0 \cup L'_1$$

where

- $A'$  the double absolute conic, and
- $L'_0, L'_1$  are two lines which might form a singular double line or a double line.

Let  $A(Y) = (\text{Pic}(X), D, \cdot, h)$  be the enhanced Picard group. Let  $B(3) = \mathbf{Z}\langle H, Q_1, Q_2, Q_3 \rangle$  be the standard Del Pezzo basis. Let  $F(Y)$  be the effective Del Pezzo zero-set .

**b)** We have that  $A(Y) \cong B(3)$  and up to Cremona equivalence we have either  $F(Y) = \emptyset$ ,  $F(Y) = \{Q_2 - Q_3\}$ , or  $F(Y) = \{Q_2 - Q_3, H - Q_1 - Q_2 - Q_3\}$ .

Let  $(C, R)$  be the strict unprojected class of a double conic in  $Z$ .

**c)** We have that  $C = 2H - Q_i - Q_j$  or  $C = 2(H - Q_i)$  for some  $i \neq j \in [1, 3]$ .

Let  $(A, F_0)$  be the strict unprojected class of  $A'$ . Let  $(B, F_1)$  be the strict unprojected class of  $L'_0 \cup L'_1$ . Let  $F = F_0 + F_1$ . Let  $W \in A(Y)$  be the unprojected class of  $W'$ .

**d)** We have that  $W = A + B + F = 3H - Q_1 - Q_2 - Q_3$ .

Let  $G(Y)$  be the irreducible Del Pezzo two-set. Let  $E(Y)$  be the Del Pezzo one-set. Let  $G_{circles}(Y) \subset G(Y)$  be the classes of families of circles.

e) If  $F(Y) = \emptyset$  then up to Cremona equivalence we have that

- $A = 2(H - Q_3)$ ,  $B = (Q_3) + (H - Q_1 - Q_2)$ ,  $F = 0$ ,
- $E(Y) = \{ Q_i, H - Q_i - Q_j \mid i \neq j \in [1, 3] \}$ ,  $G(Y) = \{ H - Q_i \mid i \in [1, 3] \}$ , and
- $G_{circles}(Y) = \{ H - Q_1, H - Q_2 \}$ .

f) If  $F(Y) = \{Q_2 - Q_3\}$  then up to Cremona equivalence we have that

- $A = 2H - Q_2 - Q_3$ ,  $B = (Q_3) + (H - Q_1 - Q_2)$ ,  $F = Q_2 - Q_3$ ,
- $E(Y) = \{ Q_1, Q_3, H - Q_1 - Q_2, H - Q_2 - Q_3 \}$ ,  $G(Y) = \{ H - Q_1, H - Q_2 \}$ , and
- $G_{circles}(Y) = \{ H - Q_1, H - Q_2 \}$ .

g) If  $F(Y) = \{Q_2 - Q_3, H - Q_1 - Q_2 - Q_3\}$  then up to Cremona equivalence we have that

- $A = 2H - Q_2 - Q_3$ ,  $B = 2Q_3$ ,  $F = (Q_2 - Q_3) + (H - Q_1 - Q_2)$ ,
- $E(Y) = \{ Q_1, Q_3 \}$ ,  $G(Y) = \{ H - Q_1, H - Q_2 \}$ , and
- $G_{circles}(Y) = \{ H - Q_1, H - Q_2 \}$ .

**Proof:**

*Claim 1:* We have that **a)**.

Assume by contradiction that the plane at infinity section  $W'$  of  $Z$  consist of the double absolute conic  $2A'$  and an irreducible conic  $C'$ . It follows that  $C'$  would be the unique conic through a generic point  $p$  on  $C'$  outside  $A'$ , which intersect  $A'$  in at least 2 points. Indeed any other conic intersects  $W'$  in  $p$  and at most one other point. We have that  $Z$  has two families of circles. ⚡ It follows this claim holds.

*Claim 3:* We have that **b)** and **d)**.

From Theorem 32 it follows that  $Y$  is a weak Del Pezzo surface. From Proposition 12 it follows that  $A(Y) \cong B(3)$  with  $D = 3H - Q_1 - Q_2 - Q_3$ . We have that  $D = W$  is the class of the pull back of hyperplane sections. From Lubbes [2012b] it follows that these are the only cases for  $F(Y)$  such that  $\#G(Y) \geq 2$ .

*Claim 4:* We have that  $F(Y)$ ,  $E(Y)$ ,  $G(Y)$  and  $G_{circles}(Y)$  as in **e)**, **f)** and **g)**.

From Proposition 17 it follows that  $\#G_{circles}(Y) = 2$ . See Lubbes [2012a] for the remaining.

*Claim 5:* We have that  $CE \geq 0$  for all  $E \in E(Y) \cup F(X)$ .

Assume by contradiction that  $CE < 0$  for some  $E \in E(Y)$ . We have that  $E$  is a fixed component of  $A$ . It follows that the absolute conic has a line as component. ⚡ From the definition of strict unprojection it follows that  $CE \geq 0$  for  $E \in F(Y)$ .

Let  $C = c_0H - c_1Q_1 - c_2Q_2 - c_3Q_3$ .

*Claim 6:* We have that **c**).

From claim 4) and claim 5) it follows that  $c_i \in \mathbf{Z}_{\geq 0}$  and  $c_0 \geq c_i + c_j$ . From  $c_0 \geq c_i + c_j$  it follows that  $3c_0 \geq 2(c_1 + c_2 + c_3)$ . From  $DA = 4$  it follows that  $3c_0 = c_1 + c_2 + c_3 + 4$ . It follows that  $c_1 + c_2 + c_3 \leq 4$ . From these Diophantine equations it follows that  $c_0 \leq 2$  and thus this claim holds.

*Claim 7:* We have that **e**).

We have that  $F = 0$ . From Proposition 46 and claim 4) it follows that  $(H - Q_1)A = (H - Q_2)A = 2$ . From  $A = C$  as in claim 6) it follows that  $A = 2(H - Q_3)$ . This claim follows from claim 3).

*Claim 8:* We have that  $B$  is the sum of two (not necessarily different) classes in  $E(Y)$ .

From Proposition 17 it follows that unprojected classes of non-singular lines are in  $E(Y)$ . Suppose by contradiction that  $B$  is not the sum of classes in  $E(Y)$ . From claim 1) it follows that  $W'$  consists of the double absolute conic and a double line. This double line has to come from the projection of a conic, and thus  $B \in G(Y)$ . From claim 4) it follows that classes in  $G(Y)$  have pairwise a moving intersection point. There are two families of circles. A conic in a family intersects the plane at infinity in at most two points. This moving intersection point with the double line of  $B$  is moving away from the absolute conic. Conics are circles if they intersect the absolute conic in two different points. ⚡

*Claim 9:* We have that **f**).

From Proposition 46 and claim 4) it follows that  $(H - Q_1)(A + F) = (H - Q_2)(A + F) = 2$ . We have that  $(H - Q_1)F = 0$ . We consider  $A = C$  as in claim 6) such that  $A(Q_2 - Q_3) \geq 0$  and  $A(H - Q_1) = 2$ . It follows that  $A \in \{2(H - Q_2), 2H - Q_2 - Q_3\}$ . From claim 8) it follows that  $B$  is the sum of two classes in  $E(Y)$ . We have that either  $(A = 2(H - Q_2), B = Q_3 + (H - Q_1 - Q_2) \text{ and } F = 2(Q_2 - Q_3))$  or  $(A = 2H - Q_2 - Q_3, B = Q_3 + (H - Q_1 - Q_2) \text{ and } F = Q_2 - Q_3)$ . From  $H - Q_2$  intersecting the absolute conic in two different points it follows that  $(H - Q_2)F < 2$ . It follows that  $F \neq 2(Q_2 - Q_3)$ .

*Claim 10:* We have that **g**).

From the definitions it follows that  $E(Y) = \{Q_1, Q_3\}$  and  $G(Y) = \{H - Q_1, H - Q_2\}$ . From Proposition 46 and  $\#G(Y) = 2$  it follows that  $H - Q_1$  and  $H - Q_2$  define families of circles. From Proposition 46 it follows that  $(H - Q_1)(A + F) = (H - Q_2)(A + F) = 2$ . We have that  $(H - Q_1)F = 0$ . From  $H - Q_2$  intersecting the absolute conic in two different points it follows that  $(H - Q_2)F < 2$ . We consider  $A = C$  as in claim 6) such that  $A(Q_2 - Q_3) \geq 0$ ,  $A(H - Q_1 - Q_2 - Q_3) \geq 0$  and  $A(H - Q_1) = 2$ . It follows that  $A \in \{2(H - Q_2), 2H - Q_2 - Q_3\}$ . From claim 8) it follows that  $B$  is the sum of two classes in  $E(Y)$ . We have that either  $(A = 2(H - Q_2), B = Q_3 + Q_3 \text{ and } F = 2(Q_2 - Q_3) + (H - Q_1 - Q_2 - Q_3))$  or  $(A = 2H - Q_2 - Q_3, B = Q_3 + Q_3 \text{ and } F = (Q_2 - Q_3) + (H - Q_1 - Q_2 - Q_3))$ . From  $H - Q_2$  intersecting the absolute conic in two different points it follows that  $(H - Q_2)F < 2$ . It follows that  $F \neq 2(Q_2 - Q_3) + (H - Q_1 - Q_2 - Q_3)$ . ☕

**Proposition 50. (properties of celestials of type (4,0))**

Let  $Y = (X, D, V^3)$  be a celestial of type (4,0). Let  $B(5) = \mathbf{Z}\langle H, Q_1, Q_2, Q_3, Q_4, Q_5 \rangle$  be the standard Del Pezzo basis.

a) We have that  $A(Y) \cong B(5)$ .

Let  $F(Y)$  be the effective Del Pezzo zero-set (see Definition 14 for short notation of its elements). Let  $G(Y)$  be the real irreducible Del Pezzo two-set (see Definition 16 for short notation of its elements). Let  $G_{circles}(Y) \subset G(Y)$  be the classes of families of circles.

b) Up to Cremona equivalence we have that  $F(Y) = \{14, 1134, 25, 1235\}$ ,  $G(Y) = \{a1, a2, a3, b3\}$  and  $G_{circles} = \{a1, a2\}$ .

Let  $Z \subset \mathbf{P}^3$  be the projected model of  $Y$ . Let  $W \in A(Y)$  be the unprojected class of the plane at infinity section of  $Z \subset \mathbf{P}^3$ .

c) We have that  $W = (14) + (1134) + (25) + (1235) + (H - Q_1 - Q_2) + (Q_3) + (Q_4) + (Q_5)$  und thus  $W$  has 4 lines as components which intersect in 4 singular double points.

**Proof:**

*Claim 1:* We have that a).

From Theorem 32 it follows that  $Y$  is a weak Del Pezzo surface of degree four. This claim follows from Proposition 12.

*Claim 2:* We have that  $G_{\mathbf{R}}(Y) \subset \{ ai, bi \mid i \in [1, 5] \}$  and  $F(Y) \subset \{ 1ijk, ij \mid i, j, k \in [1, 5] \}$ .

See Lubbes [2012a].

Let  $noc(\mathbf{C})$  be the number of components of the intersection diagram of  $F(Y)$  (thus the number of summands in the Dynkin type).

*Claim 3:* We have that b).

From Proposition 17 it follows that classes in  $G(Y)$  define families of conics. The absolute conic is not contained in  $Z$ . A conic is a circle if it intersects the absolute conic in two different points. It follows that families of circles must have base points on the absolute conic. From Proposition 17 it follows that these base points are determined by  $F(Y)$ . We have that base points are singular points. From Table 19 it follows that  $noc(\mathbf{C}) \leq 4$ . We need at least two different base points on the absolute conic thus  $noc(\mathbf{C}) \geq 2$ . From claim 2) it follows that there exists no  $G \neq G' \in G_{\mathbf{R}}(Y)$  and  $F_1, F_2 \in F(Y)$  such that  $(F_1 F_2 = 0)$  and  $(GF_1 > 0, GF_2 > 0)$  and  $(G'F_1 > 0, G'F_2 > 0)$ . From claim 2) it follows that there exists no  $G \neq G' \in G_{\mathbf{R}}(Y)$  and  $F_1, F_2, F_3 \in F(Y)$  such that  $(F_1 F_2 = F_1 F_3 = F_2 F_3 = 0)$  and  $(GF_1 > 0, GF_2 > 0)$  and  $(G'F_2 > 0, G'F_3 > 0)$ . It follows that  $noc(\mathbf{C}) = 4$  with Dynkin type  $4A_1$ . From Proposition 18 it follows that up to Cremona equivalence we may assume that  $F(Y) = \{14, 1134, 25, 1235\}$ . It follows that  $G(Y) = \{a1, a2, a3, b3\}$ .

*Claim 4:* We have that c).

From claim 1) it follows that  $W = D = 3H - Q_1 - \dots - Q_5$  is the class of the pull back of

hyperplane sections. We have that  $D - (14) + (1134) + (25) + (1235) = 2(H - Q_3 - Q_4 - Q_5)$ . If  $2(H - Q_3 - Q_4 - Q_5)E < 0$  for some  $E$  in  $E(Y)$  then  $E$  must be a fixed component of  $2(H - Q_3 - Q_4 - Q_5)$ . It follows that  $Q_3, Q_4$  and  $Q_5$  are unprojected classes of lines. The remainder class  $H - Q_1 - Q_2$  is also the class of a line. The unprojected classes of the lines do not intersect. It follows that the lines intersect on  $Z$  in the singular locus. ☕

**Proposition 51. (singular locus of embedded projections of celestials)**

Let  $Y = (X, D, V^3)$  be a celestial. Let  $Z \subset \mathbf{P}^3$  be the embedded projection of  $Y$ . Let  $d \geq 3$  be the degree of  $Z$ . Let  $\text{sng}Z$  be the singular locus of  $Z$ .

a) We have that  $\text{sng}Z$  is contained in the intersection of  $Z$  with a hypersurface of degree  $d - 3$ .

Let  $S_i \subset \text{sng}Z$  for  $i \in I$  be the curve components of  $\text{sng}Z$  (thus not the isolated points). Let  $d_i \in \mathbf{Z}_{\geq 1}$  be the degree of  $S_i$ . Let  $m_i \in \mathbf{Z}_{\geq 2}$  be the multiplicity of  $S_i$ .

b) We have that  $\sum_{i \in I} d_i m_i (m_i - 1) \leq (d - 1)(d - 2) - 2$ .

**Proof:** Since we want to work with singular surfaces we need to use the notion of module sheaves. For the proof we assume that the reader is familiar with [Hartshorne \[1977\]](#). Let  $\mathcal{O}$  be a ring sheaf. Let  $A$  and  $B$  be  $\mathcal{O}$ -module sheaves of finite rank. Let  $h^0(A)$  be the dimension over the ground field of the global sections (thus the 0th sheaf cohomology). Let  $\mathcal{O}_Z(i)$  be the structure sheaf of  $Z$  twisted by  $i \in \mathbf{Z}$  (see [Hartshorne \[1977\]](#), chapter 2, section 5, page 117).

*Claim 1:* We have that  $\varphi_* A \otimes B = \varphi_*(A \otimes \varphi^* B)$ ,  $h^0(\varphi_* A) = h^0(A)$  and  $\varphi^* \mathcal{O}_Z(1) = \mathcal{O}_X(1)$ .

The first statement is the projection formula (see [Hartshorne \[1977\]](#), chapter 2, section 5, exercise 1, page 124). The second statement follows from the definition of the pushforward of a sheaf. The third statement follows from  $\mathcal{O}_Z(1)$  representing the hyperplane sections and  $\varphi^* \mathcal{O}_Z(1) = (\varphi_D \circ \pi)^* \mathcal{O}_Z(1)$ .

Let  $\varphi := \varphi_{V^3}$  be the map associated to  $V^3 \subset \mathbf{P}(H^0(X, D))$ . Let  $W_X$  be the canonical sheaf of  $X$  (see [Hartshorne \[1977\]](#), chapter 2, section 8, page 180). Let  $W_Z$  be the canonical sheaf of  $Z$  (thus  $W_Z := \varphi_* W_X$ ). Let  $W_Z^\circ$  be the dualizing sheaf of  $Z$  (see [Hartshorne \[1977\]](#), chapter 3, section 7, page 241).

*Claim 2:* We have that  $W_Z \otimes \mathcal{O}_Z(1) \subset \mathcal{O}_Z(d - 3)$ .

From the canonical sheaf formula for hypersurfaces it follows that  $W_Z^\circ = \mathcal{O}_Z(d - 3 - 1)$  (see [Hartshorne \[1977\]](#), chapter 2, section 8, example 20.3, page 183). From differential forms vanishing at the singularities it follows that  $W_Z \subset W_Z^\circ$ . From tensoring with an invertible sheaf being left exact it follows that this claim holds.

Let  $K$  be the canonical divisor class of  $X$ .

*Claim 3:* We have that  $h^0(W_Z \otimes \mathcal{O}_Z(1)) = h^0(K + D)$ .

From  $W_Z = \varphi_* W_X$  it follows that  $h^0(\varphi_* W_X \otimes \mathcal{O}_Z(1)) = h^0(W_Z \otimes \mathcal{O}_Z(1))$ . From claim

1) it follows that  $h^0(\varphi_* W_X \otimes O_Z(1)) = h^0(\varphi_*(W_X \otimes \varphi^* O_Z(1)))$ . From claim 1) it follows that  $h^0(\varphi_*(W_X \otimes \varphi^* O_Z(1))) = h^0(W_X \otimes \varphi^* O_Z(1))$ . From claim 1) it follows that  $h^0(W_X \otimes \varphi^* O_Z(1)) = h^0(W_X \otimes O_X(1))$ . On  $X$  we can represent sheaves by divisor classes (see [Hartshorne \[1977\]](#), chapter 2, section 6, corollary 16, page 145). It follows that  $h^0(W_X \otimes O_X(1)) = h^0(K + D)$ .

*Claim 4:* We have that  $h^0(K + D) = 1$ .

From Theorem 32 it follows that  $Y$  is a weak Del Pezzo surface. From  $d \geq 3$  it follows that  $D = -K$  and thus this claim holds.

*Claim 5:* We have that **a**).


From claim 2) it follows that the singular locus is the base locus of a linear series of hypersurface sections of degree  $d - 3$ . From claim 3) and claim 4) it follows that the singular locus is contained in a fixed hypersurface section in this linear series.

Let  $p_a(D)$  be the arithmetic genus of  $D$ . Let  $H$  be a generic hyperplane section of  $Z$ . Let  $p_g(H)$  be the geometric genus of  $H$ . Let  $\delta_p H$  be the delta invariant at a point  $p \in H$ .

*Claim 6:* We have that **b**)

From the adjunction formula it follows that  $p_a(D) = \frac{1}{2}D(D + K) + 1 = 1$ . From Bertini theorem it follows that  $p_a(D) = p_g(D)$ . The divisors in the linear series  $|D|$  are birationally mapped to hyperplane sections of  $Z$ . It follows that  $p_g(H) = p_a(D) = 1$ . From the genus formula for plane curves it follows that  $p_g(H) = \frac{1}{2}(d - 1)(d - 2) - \sum_{p \in H} \delta_p H$ . A hyperplane section has a singularities at the intersections with  $S_i$  for  $i \in I$  and misses the isolated singular points. A singular point of the birational projection of a smooth curve with  $m_i$  points as preimage defines a lower bound on the delta invariant of this singularity. We have that  $\delta_p H \geq \frac{1}{2}m_i(m_i - 1)$  for  $d_i$  points  $p \in S_i$ . It follows that  $\sum_{p \in H} \delta_p H \geq \frac{1}{2} \sum_{i \in I} d_i m_i (m_i - 1)$ .

It follows that  $\frac{1}{2} \sum_{i \in I} d_i m_i (m_i - 1) \leq \frac{1}{2}(d - 1)(d - 2) - 1$ .

For example it can happen that  $m_i = \delta_p H = 2$  for  $p \in S_i$ . 

**Proposition 52. (singular locus of octic Möbius models of celestials)**

Let  $Y = (X, D, V^3)$  be a celestial of Möbius degree 8. Let  $Z \subset \mathbf{P}^3$  be the projected model of  $Y$ . Let  $\text{MTD}(\mu, f, \beta)$  be the Möbius transformation diagram. Let  $M$  be the Möbius model of  $Y$ .

- a)** We have that  $M \xrightarrow{f} Z$  is an isomorphism outside the tangent hyperspace section at the center of projection of  $M$ , and the plane at infinity section of  $Z$
- b)** The singular locus of  $M$  is a curve of at most degree eight.
- c)** There are no isolated singularities on  $M$ .
- d)** There are no non-singular lines on  $M$ .



- e) Points of  $M$  have multiplicity either 1, 2 or 4.
- f) A point in  $M$  of multiplicity 4 is the intersection of 4 double lines.
- g) There are no 3 lines through a point in  $M$ .

**Proof:**

*Claim 1:* We have that **a**).

The birational map  $f$  is almost everywhere an isomorphism. We have that  $f$  is not defined at the center of projection  $\infty = (0 : 0 : 0 : 1 : 1)$ . From Proposition 27 it follows that  $\mathbf{P}^3 \xrightarrow{f^{-1}} S^3$  is not defined at the absolute conic. If the center of projection lies on  $M$  then it is blown up to an exceptional curve in the plane at infinity section of  $Z$ .

*Claim 2:* We have that **b**).

Up to Möbius equivalence we may assume that  $Z$  is of type  $(8, 4)$ . Curves of degree  $k$  on  $Z$  intersect the plane at infinity, and thus the absolute conic, in  $k$  points. From Proposition 28 it follows that  $M \xrightarrow{f} Z$  preserves the degree of the singular locus outside the absolute conic. This claim follows from Proposition 51.

*Claim 3:* We have that **c**) and **d**).

Up to Möbius equivalence we may assume that  $Z$  is of type  $(8, 4)$ . This claim follows from claim 1) and Proposition 47.

*Claim 4:* We have that **e**).

Assume by contradiction that  $M$  contains point of multiplicity  $m \notin \{1, 2, 4\}$ . From Proposition 29 it follows that  $Z$  is Möbius equivalent to a celestial of type  $(8 - m, 4 - m)$ . From Theorem 32 it follows that no such surface exists. ⚡

Let  $\text{MTD}(\mu, f, \beta)$  be the Möbius transformation diagram. Let  $Z_i \subset \mathbf{P}^3$  be the projection of  $M$  from a point with multiplicity  $8 - i$ .

*Claim 5:* We have that **f**).

From Proposition 52 it follows that  $M$  has no isolated double points. From claim 1) it follows that  $Z_4$  is of type  $(4, 0)$ . From Proposition 50 it follows that  $Z_4$  has 4 isolated double points at the plane at infinity section. These double points can only come from 4 lines through the center of projection.

*Claim 6:* We have that **g**).

Assume by contradiction that  $M$  contains three double lines through a point  $q$ . From claim 4) and claim 5) it follows that  $q$  is of multiplicity 2. Each of the lines is projected to an isolated singular point on  $Z_6$ . From Proposition 49 it follows that there are at most 2 isolated singular points. ⚡ ☕

**Proposition 53. (real structures of celestials of Möbius degree 8)**

Let  $Y = (X, D, V^3, \sigma)$  be a real celestial of type  $(d, c)$  such that  $d - c = 4$ . Let  $A(Y) = (\text{Pic}(X), D, \cdot, h, \sigma)$  be the real enhanced Picard group. Let  $F(Y)$  be the effective Del Pezzo zero-set.

**a)** If  $(d, c) \in \{(8, 4), (7, 3), (6, 2)\}$  then  $F(Y) = \emptyset$  and the real structure  $A(Y) \xrightarrow{\sigma} A(Y)$  is the identity.

Let  $G_{\mathbf{R}}(Y)$  be the real irreducible Del Pezzo two-set. We use the notation in Definition 14 and Definition 16 for elements in  $F(Y)$  and  $G_{\mathbf{R}}(Y)$ .

**b)** If  $(d, c) = (4, 0)$  then up to real Cremona equivalence  $A(Y) \xrightarrow{\sigma} A(Y)$  is defined by  $RI = 13$  in Table 20 such that  $\sigma(14) = 1235$ ,  $\sigma(25) = 1134$  and  $G_{\mathbf{R}}(Y) = \{a1, a2, a3, b3\}$ .

**Proof:** Let  $Z \subset \mathbf{P}^3$  be the projected model of  $Y$ .

*Claim 1:* We have that either  $F(Y)$  is permuted by the real structure or  $F(Y) = \emptyset$ .

From Proposition 52 it follows that the Möbius model of  $Y$  has no isolated singularities. The Möbius model is contained in  $S^3$  and thus does not contain real lines. Isolated singularities on  $Z$  can only come from complex lines, which go through the center of projection. It follows that  $F(Y)$  is permuted by the real structure.

*Claim 2:* We have that **a)**.

See Lubbes [2012a] for the classification of real structures up to Cremona equivalence. The identity map is the only real structure such that  $\#G_{\mathbf{R}}(Y) \geq 2$ . From claim 1) it follows that  $F(Y) = \emptyset$ .

*Claim 3:* We have that **b)**.

We have that  $a1$  is a family of circles through base points 14 and 1235. We have that  $a2$  is a family of circles through base points 25 and 1134. From claim 1) it follows that  $\sigma(14) = 1235$  and  $\sigma(25) = 1134$ . It follows that the real structure  $\sigma$  is defined by  $RI = 13$  in Table 20. It follows that this claim holds. ☕

The following theorem essentially extends Proposition 52 by also taking the real structure into account.

**Theorem 54. (celestial of Möbius degree 8)**

Let  $Y = (X, D, V^3, \sigma)$  be a real celestial of Möbius degree 8. Let  $M$  be the Möbius model of  $Y$ .

- a)** We have that  $Y$  is Möbius equivalent to celestial of types  $(4, 0)$ ,  $(6, 2)$ ,  $(7, 3)$  or  $(8, 4)$ .
- b)** All lines in  $M$  are complex double lines. If exactly 2 lines intersect then there are at least 4 conjugate lines.
- c)** If  $M$  has a point  $p$  of multiplicity 4 then  $p$  is the intersection of 4 conjugate double lines. The remaining singular locus of  $M$  consists of either one of the following components:

- two co-spherical double circles intersecting at  $p$ , or
- one double circle outside  $p$  intersecting two conjugate double lines.

**Proof:**

*Claim 1:* We have that **a**).

This claim follows from Theorem 32 and Proposition 29.

*Claim 2:* Lines in  $M$  are non-real double lines.

There are no real lines in  $S^3$ . This claim follows from Proposition 52.

Let  $\text{MTD}(\mu, f, \beta)$  be the Möbius transformation diagram. Let  $Z_i \subset \mathbf{P}^3$  be the projection of  $M$  from a point with multiplicity  $8 - i$ .

*Claim 3:* The intersection of exactly 2 lines in  $M$  is a non-real double point.

Assume by contradiction that the intersection of the 2 lines is a real point. From Proposition 52 it follows that a point of multiplicity 4 is the intersection of 4 lines. It follows that the intersection of the 2 double lines is a point of multiplicity 2. We have that the projection  $Z_6$  from this real point results in a real celestial with 2 isolated singular points. From Proposition 53 it follows that the real structure of  $Z_6$  is the identity and that there are no conjugate isolated singularities.  $\nexists$

*Claim 4:* If exactly 2 lines intersect then there are at least 4 conjugate lines.


From claim 3) it follows that the lines are not conjugate. Since the singular locus is real we have that each line is conjugate to another line.

*Claim 5:* We have that **b**)

This claim follows claim 2), claim 3) and claim 4).

Let  $\text{sng}Z_4$  be the singular locus of  $Z_4$ .

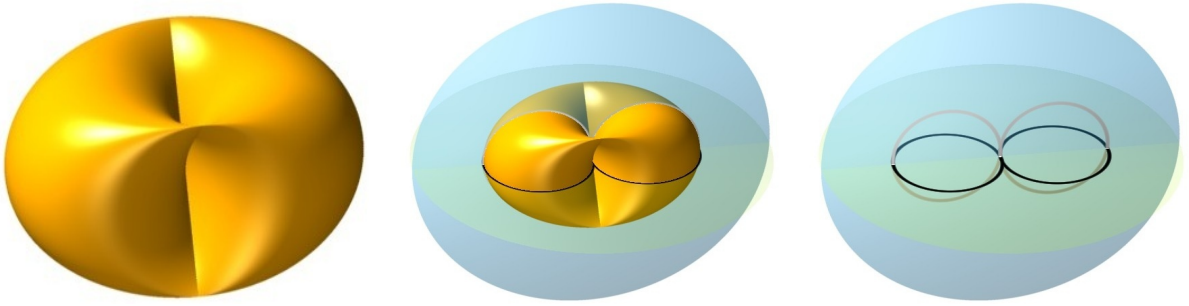
*Claim 6:* We have that **c**).

From Proposition 50 it follows that  $\text{sng}Z_4$  contains 4 isolated double points which lie on the absolute conic. From Proposition 51 it follows that the singular locus of  $Z_4$  contains a real planar curve of degree two. It follows that the one dimensional components of  $\text{sng}Z_4$  consist of 2 double lines, or a real double circle. From claim 4) it follows that  $M$  has no planar section with exactly two lines. It follows that if  $\text{sng}Z_4$  is reducible then 2 co-spherical double circles in  $M$  are projected to 2 double lines in  $\text{sng}Z$ . From Proposition 52 it follows that  $M \xrightarrow{f} Z_4$  is an isomorphism almost everywhere. If the one dimensional component of  $\text{sng}Z_4$  is a circle then it intersects the plane at infinity at two complex conjugate points. From Proposition 50 it follows that these are isolated double points. It follows that  $\text{sng}M$  consists of a double circle which intersects two conjugate double lines. 

## 12 Examples of celestials of Möbius degree 8

### Example 55. (celestial of type (4,0))

Let us consider a surface which is formed by moving one circle along another circle, in such a way that all circles are parallel to each other. The author would like to thank Helmut Pottmann for making me aware of this surface. Images of this surface can be found in for example [Pottmann et al. \[2007\]](#). We will consider the case where the two families of circles have the same radius. Each of the two plane sections in the picture below shows two circles in one family.



The parametrization of this surface is:

$$\gamma : [0, 2\pi]^2 \rightarrow \mathbf{R}^3, \quad (\alpha, \beta) \mapsto (\sin(\alpha) - \cos(\beta), \cos(\alpha), \sin(\beta)).$$

The planes in the image above are defined by the equations  $y = 0$  and  $z = 0$ . From the trigonometric identities it follows that after homogenization this surface can be defined as:

$$Z : F(x : y : z : w) = (x^2 + y^2 + z^2 - 2w^2)^2 - 2(w^2 - y^2)(w^2 - z^2) = 0 \subset \mathbf{P}^3.$$

Let  $\text{sng}Z$  be the singular locus of  $Z$ .

We have that  $\text{sng}Z$  consists of two real lines parametrized by:

$$\rho_{\pm} : \mathbf{P} \rightarrow Z, \quad (s : t) \mapsto (0 : t : \pm t : s) \subset \text{sng}Z.$$

and 4 conjugate points which lie on the absolute conic:  $(1 : 0 : \pm i : 0)$  and  $(1 : \pm i : 0 : 0)$ .

We know from Theorem 32 that  $Z$  must be a weak Del Pezzo surface. From  $Z$  being a degree 4 surface with two double lines, it follows that the geometric genus of the generic hyperplane section is indeed one. We find that this surface is of type (4, 0).

The planes spanned by a parallel family of circles are the planes through the line spanned by either  $(1 : 0 : \pm i : 0)$  or  $(1 : \pm i : 0 : 0)$ .

From  $F(x : y : z : 0) = (x - iy + iz)(x - iy - iz)(x + iy + iz)(x + iy - iz)$  it follows that the plane at infinity section of  $Z$  factors into four lines. We predicted this in Proposition 50.

**Example 56. (Möbius model of celestial of type (4,0))** Let  $\text{MTD}(\mu, f, \beta)$  be the Möbius transformation diagram. Let  $Z \subset \mathbf{P}^3$  be the celestial of type (4, 0) as defined in the previous Example 55. Let  $S(a : b : c : d : e) = a^2 + b^2 + c^2 + d^2 - e^2 = 0 \subset \mathbf{P}^4$ . Let  $M(a : b : c : d : e) = 4b^2c^2 - 4b^2d^2 + 8b^2de - 4b^2e^2 - 4c^2d^2 + 8c^2de - 4c^2e^2 - 5d^4 + 8d^3e + 2d^2e^2 - 8de^3 + 3e^4$ .

We have that  $M : S = M = 0 \subset S^3 \subset \mathbf{P}^4$  is the Möbius model of  $Z$ .

The singular locus of  $M$  is determined by the points where the Jacobian of  $[S, M]$  has rank 1 or lower.

We find the following four double lines lying in the plane  $d - e = 0$ :

$$\rho_{\pm} : \mathbf{P} \rightarrow M, \quad (s : t) \mapsto (s : \pm is : 0 : t : t) \subset \text{sing}M$$

and

$$\rho'_{\pm} : \mathbf{P} \rightarrow M, \quad (s : t) \mapsto (s : 0 : \pm is : t : t) \subset \text{sing}M.$$

We also find two real double conics defined by

$$C_+ : a = b + c = 2c^2 + d^2 - e^2 = 0 \subset \text{sing}M \text{ and } C_- : a = b - c = 2c^2 + d^2 - e^2 = 0 \subset \text{sing}M.$$

To check that these conics are in the singular locus we first polynomial reduce the entries of the Jacobian with the ideal of each curve. We find that the rank is indeed 1.

The two double conics  $C_{\pm}$  go through the center of projection  $(0 : 0 : 0 : 0 : 1 : 1)$ . We find that these circles also intersect in the point  $(0 : 0 : 0 : 1 : -1)$ .

From Theorem 54 it follows that the singular locus consists of 4 double lines and 2 double conics. The 4 double lines contract to the singular points on  $Z$ , which lie on the absolute conic. The 2 double conics contract to 2 double lines on  $Z$ .

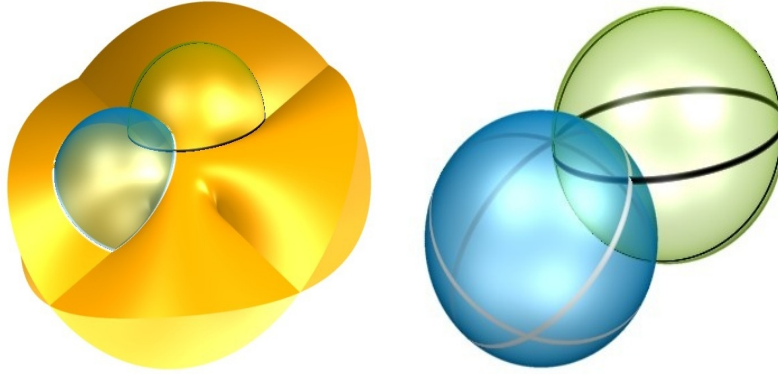
The projection  $M \subset \mathbf{P}^4 \xrightarrow{f} Z \subset \mathbf{P}^3$  results in a surface of degree 4. It follows that the pullback of hyperplane sections along  $f$  are curves with a point of multiplicity 4 at the center of projection.

**Example 57. (celestial of type (6,2))** Let  $Z \subset \mathbf{P}^3$  be the celestial of type  $(4,0)$  as defined in Example 55. Let  $\mu_6$  as in Definition 38.

We have that

$$Z' := \mu_6(Z) : w^2(x^2 + y^2 + z^2 - 2yz)(x^2 + y^2 + z^2 + 2yz) - 4x^2(x^2 + y^2 + z^2)^2 = 0 \subset \mathbf{P}^3.$$

We recall from Example 55 that the two families of circles of  $Z$  can be defined by the families of hyperplane sections  $y - tw = 0$  and  $z - tw = 0$  for  $t \in \mathbf{C}$ . We take the image of these families of hyperplane sections with respect to  $\mu$  and obtain the families of spheres  $t(x^2 + y^2 + z^2) + y = 0$  respectively  $t(x^2 + y^2 + z^2) + z = 0$  for  $t \in \mathbf{C}$ . Below we show a spherical section of  $Z'$  for each of the two families of circles for some fixed  $t \in \mathbf{R}$ . On the right are enlarged versions of the two spheres intersected with  $Z'$ .



Let  $\text{sng}Z'$  be the singular locus of  $Z'$ .

The absolute conic is contained in the singular locus with multiplicity two. We find 4 complex double lines:

$$\rho_{\pm} : \mathbf{P} \rightarrow Z', \quad (s : t) \mapsto (s : \pm is : 0 : t) \subset \text{sng}Z',$$

and

$$\rho'_{\pm} : \mathbf{P} \rightarrow Z', \quad (s : t) \mapsto (s : 0 : \pm is : t) \subset \text{sng}Z'.$$

We also find 3 real double lines:

$$\lambda_{\pm} : \mathbf{P} \rightarrow Z', \quad (s : t) \mapsto (0 : s : \pm s : t) \subset \text{sng}Z',$$

and

$$\kappa : \mathbf{P} \rightarrow Z', \quad (s : t) \mapsto (0 : s : t : 0) \subset \text{sng}Z'.$$

So indeed the geometric genus of the hyperplane sections is one.

The surface  $Z'$  is a projection from the point  $Q$  as in Theorem 54. We find that  $Q$  is blown up to a singular double line  $\kappa$ . There are four non-singular lines in  $Z'$ :

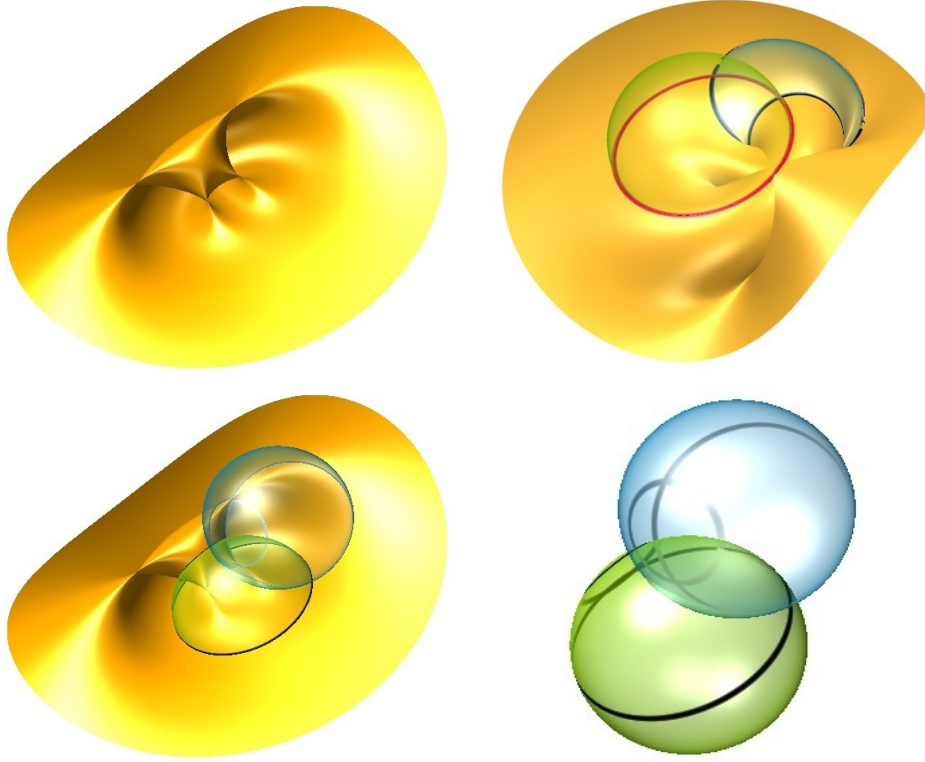
$$y = w \pm 2x = 0 \quad \text{and} \quad z = w \pm 2x = 0.$$

The reason is that there are four non-singular circles through  $Q$ .

**Example 58. (celestial of type (7,3))** Let  $Z \subset \mathbf{P}^3$  be the celestial of type (4,0) as defined in Example 55. Let  $\mu_9$  as in Definition 38.

We have that  $Z' := \overline{\mu_9(Z)} : w(4x^6 + 8x^5y + 4x^5w + 16x^4y^2 + 4x^4yw + 8x^4z^2 + x^4w^2 + 16x^3y^3 + 8x^3y^2w + 16x^3yz^2 + 8x^3z^2w + 20x^2y^4 + 8x^2y^3w + 24x^2y^2z^2 + 2x^2y^2w^2 + 4x^2z^4 + 2x^2z^2w^2 + 8xy^5 + 4xy^4w + 16xy^3z^2 + 8xy^2z^2w + 8xyz^4 + 4xz^4w + 8y^6 + 4y^5w + 16y^4z^2 + y^4w^2 + 8y^2z^4 - 2y^2z^2w^2 - 4yz^4w + z^4w^2) + 8y(x^2 + y^2 + z^2)^3 = 0$  is a celestial of type (7,3).

The upper two images are top and bottom view of  $Z'$ . Each sphere cuts out two circles in one family. Similar as in Example 57 each family of circles is defined by respectively the spherical sections  $t(x^2 + y^2 + z^2) + y = 0$  and  $t(x^2 + y^2 + z^2) + z = 0$  for  $t \in \mathbf{R}$ .

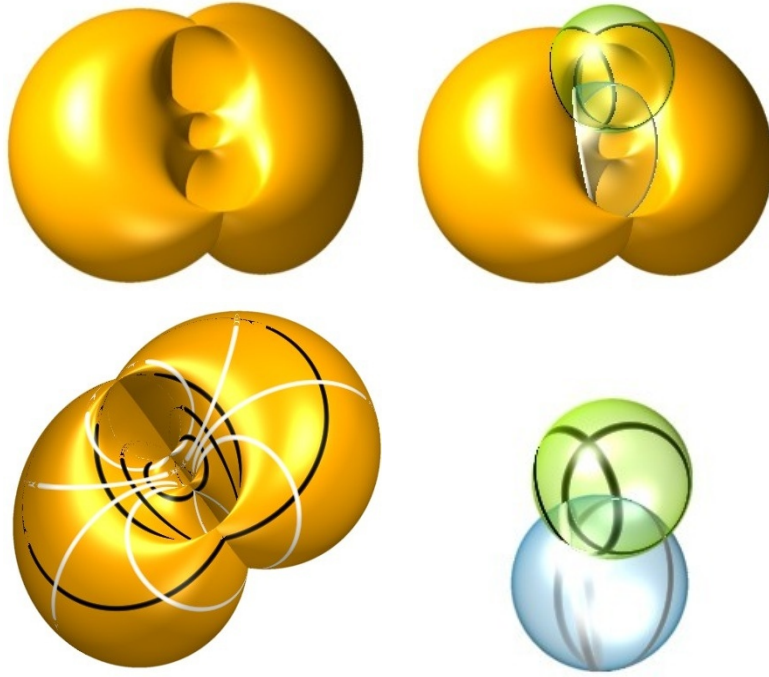


**Example 59. (celestial of type (8,4))** Let  $Z \subset \mathbf{P}^3$  be the celestial of type (4,0) as defined in Example 55. Let  $\mu_5$  as in Definition 38.

We have that  $Z' := \overline{\mu_5(Z)} : w^2(12x^6 + 36x^4y^2 + 28x^4z^2 - 64x^4zw + 26x^4w^2 + 36x^2y^4 + 56x^2y^2z^2 - 64x^2y^2zw - 12x^2y^2w^2 + 20x^2z^4 - 64x^2z^3w + 100x^2z^2w^2 - 64x^2zw^3 + 12x^2w^4 + 12y^6 + 28y^4z^2 - 38y^4w^2 + 20y^2z^4 - 28y^2z^2w^2 + 12y^2w^4 + 4z^6 - 6z^4w^2 + 4z^2w^4 - w^6) - (x^2 + y^2 + z^2)^4$  is a celestial of type (8,4).

Each sphere in the image below cuts out two circles in one family. Similar as in Example 57 each family of circles is defined by respectively the spherical sections  $t(x^2 + y^2 + z^2) + y = 0$  and  $t(x^2 + y^2 + z^2 - 2z + 1) - y = 0$  and  $(t - 1)(x^2 + y^2 + z^2) - 2tz + (t + 1) = 0$  for  $t \in \mathbf{R}$ .





**Remark 60. (Clifford translational surfaces)** A *Clifford translational surface* in  $S^3$  can be constructed by left (or right) Clifford translating a curve along another curve. Clifford translational surfaces with circles as profile curves are *celestials*. If both profile curves are great circles, this is known as a *Clifford surface* (see [Berger \[2009\]](#), chapter 18, section 8 and [Coxeter \[1998\]](#), chapter 7).

I would like to thank Helmut Pottmann for his contribution to these recent circle of ideas.

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The computations were done using the computer algebra system Sage ([Stein et al. \[2012\]](#)) with additional interfacing to Magma ([Bosma et al. \[1997\]](#)). The images were made using Surfex ([Holzer and Labs \[2008\]](#)).

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